Assignment MAT 665, Swift

H1. Sketch the phase portrait of $\vec{x}' = A\vec{x}$, starting with the phase portrait in the $\vec{y}$-coordinates

$A = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix}$. $\tau = 0$, $\delta = -4 + 20 = 16$, $\lambda = \pm 4i$

$(A - 4iI)\vec{w} = \vec{0}$ $\quad \begin{bmatrix} 2-4i & 4 \\ -5 & -2-4i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let $\vec{w} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$(2-4i)\alpha + 4\beta = 0$

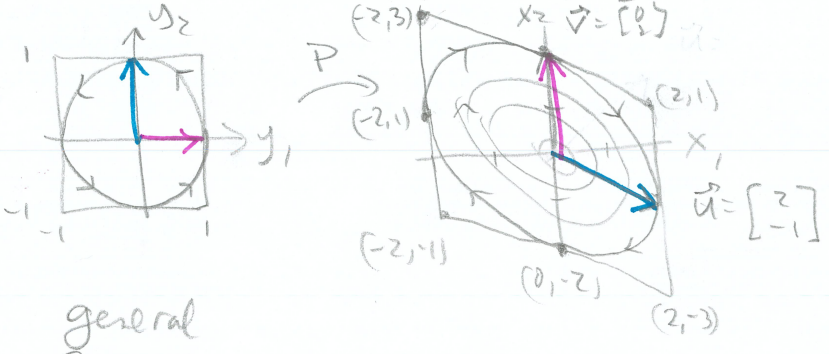
Don't worry about the 2nd equation.

$4\beta = (-2+4i)\alpha$

$\beta = \frac{-1+2i}{2}\alpha$. Let $\alpha = 2$, $\beta = -1+2i$

So $\vec{w} = \begin{bmatrix} 2 \\ -1+2i \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, so $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}$



general solution is $\vec{x}(t) = r_0\left(\cos(4t+\theta_0)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin(4t+\theta_0)\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$

Sketch one trajectory, the ellipse that touches the centers of the edges of the parallelogram. Then shrink or stretch about the origin (i.e., vary r_0) to get several other trajectories.

Note: A different scan of these solutions is on the website with the link "darker black and white solutions to H1 and H2". The document you are now reading uses color (**u** is blue), which is helpful for understanding the effect of P. You do not need colored pens on the exam if you do it correctly, but they might help.

H2. Same question, with $A = \begin{bmatrix} 1 & 4 \\ -5 & -3 \end{bmatrix}$

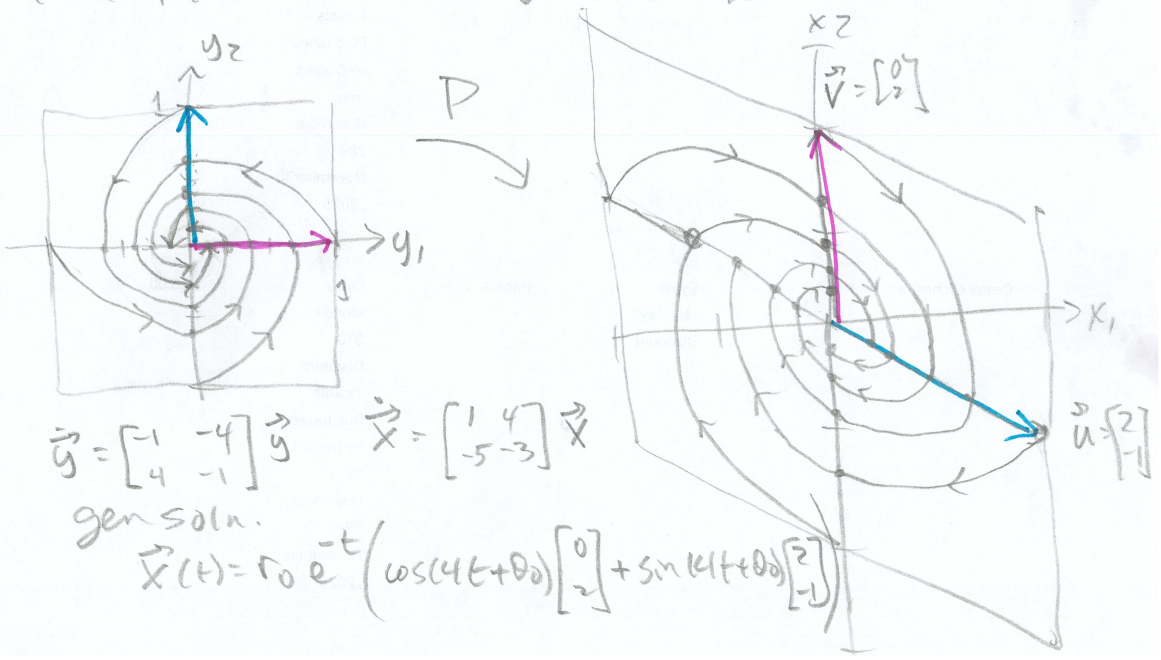
Note that $\underbrace{\begin{bmatrix} 1 & 4 \\ -5 & -3 \end{bmatrix}}_{\text{New } A} = \underbrace{\begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix}}_{\text{old } A} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

So the eigenvalues of the new A are $\lambda = -1 \pm 4i$,
and the eigenvectors are the same: $\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

The trajectories in the $\vec{y}$ coordinate are
logarithmic spirals, $r = r_0 e^{-\frac{1}{4}(\theta - \theta_0)}$.

I choose to draw 4 trajectories: $r_0 = 1$, and $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

Quarter turn: $\Delta\theta = \frac{\pi}{2}$, gives $e^{-\frac{\pi}{8}} \approx 0.68$ compression factor
half turn $\Delta\theta = \pi$, gives $e^{-\frac{\pi}{4}} \approx 0.46$ "
3 quarter turn $\Delta\theta = \frac{3\pi}{2}$, gives $e^{-\frac{3\pi}{8}} \approx 0.31$ "
full turn $\Delta\theta = 2\pi$ gives $e^{-\frac{\pi}{2}} \simeq 0.21$ "

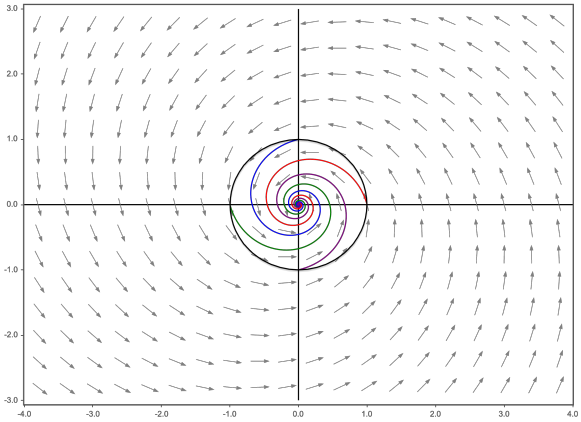


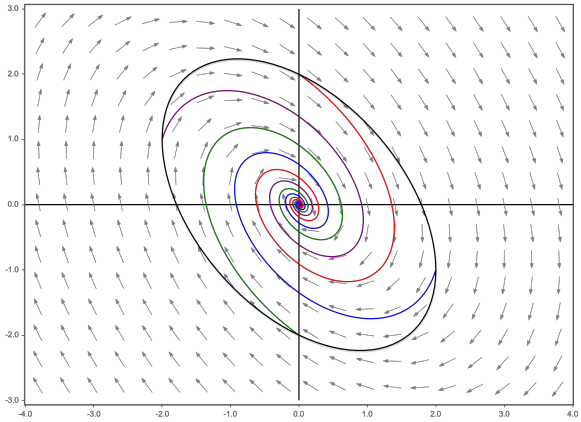
$\vec{y} = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix} \vec{y}$ $\quad \vec{x} = \begin{bmatrix} 1 & 4 \\ -5 & -3 \end{bmatrix} \vec{x}$

gen soln.

$$\vec{x}(t) = r_0 e^{-t}\left(\cos(4t + \theta_0)\begin{bmatrix} 0 \\ -2 \end{bmatrix} + \sin(4t + \theta_0)\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$$

Here are the phase portraits for H2 made with the JODE app.



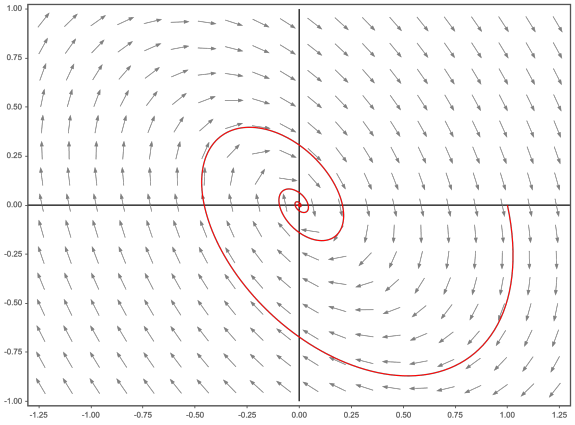


I have to admit that it was quite tedious and time-consuming to sketch the 4 different trajectories by hand! On the exam, I will not have you do anything as complicated as H2. I will probably have you draw a phase portrait for a matrix with pure imaginary eigenvalues, like H1. For pure imaginary eigenvalues I want a fairly accurate drawing of the trajectory, which is an ellipse.

For the eigenvalues of the form $\lambda = a \pm ib$, with $a, b \neq 0$, I might ask something like this:

1. Let $A = \begin{bmatrix} 1 & 4 \\ -5 & -3 \end{bmatrix}$. Find the solution to $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = (1, 0)$. Make a rough sketch of the trajectory for $t \geq 0$, using the fact that $e^{-\pi/2} \approx 0.21$. When sketching the trajectory, focus mainly on the direction of motion, and indicate the coordinates of the point where the trajectory first touches the positive x_1 axis with $t > 0$. Do not worry too much about the shape of the trajectory.

Here's the computer-generated trajectory:



As I said in class, I will not give you something like problem 1, where the solution has fractions (unless I give you ample warning!). On the other hand, I might give you the worst-case scenario:

2. Solve the initial value problem $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \mathbf{x}$, $\mathbf{x}(0) = \mathbf{x_0}$. Beware that the solution involves fractions and/or square roots. Do *not* sketch the phase portrait.

I will not ask you questions from MAT 239 review. Review Assignments 2-6.

Groundrules for the exam on Friday, March 6.

You can bring a page of handwritten notes with writing on both sides, but no book, no calculators, and no phones or other internet devices.