Assignment 6, MAT 665, Swift

Perko's Set 6, pg. 32.

1. Solve the IVP $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ with

$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$. Use Swift Method (7) and Perko Method (4).

$\tau = 4$, so $\frac{\tau}{2} = 2$, $\delta = 3 - (-2) = 5$, $\lambda = 2 \pm \sqrt{2^2 - 5}$

$\boxed{\lambda = 2 \pm i}$

(7). $A - aI = A - 2I = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$a = 2 \quad b = 1$

$$\vec{x}(t) = e^{2t}\left(\cos(t)\, I + \sin(t) \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\right)\vec{x}_0$$

$$\boxed{\vec{x}(t) = e^{2t}\begin{bmatrix} \cos(t)+\sin(t) & -2\sin(t) \\ \sin(t) & \cos(t)-\sin(t) \end{bmatrix}\vec{x}_0}$$

7 or 2mm.

(4) $A - \lambda I = A - (2+i)I = \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}$

$(A - \lambda I)\vec{w} = \vec{0}$.

Let $\vec{w} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, so $\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(1-i)\alpha - 2\beta = 0$

$\alpha + (-1-i)\beta = 0$

use 2nd one:

$\alpha = (1+i)\beta$ choose $\beta = 1$

$\vec{w} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

I cont. Note: many choices are possible.

According to my handout $\vec{u} = \frac{1}{b}(A - aI_2)\vec{v}$

So, for any $\vec{v}$, we have $\vec{u} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\vec{v}$

(I can grade any choice of eigenvector.)

So $P = [\vec{v}\ \vec{u}] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Note: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ can be used without justification

So $\vec{x}(t) = e^{2t}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\vec{x}_0$

$= e^{2t}\begin{bmatrix} \cos(t)+\sin(t) & -\sin(t)+\cos(t) \\ \sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\vec{x}_0$

$= e^{2t}\begin{bmatrix} \cos(t)+\sin(t) & -\cos(t)-\sin(t)-\sin(t)+\cos(t) \\ \sin(t) & -\sin(t)+\cos(t) \end{bmatrix}\vec{x}_0$

$$\vec{x}(t) = e^{2t}\begin{bmatrix} \cos(t)+\sin(t) & -2\sin(t) \\ \sin(t) & \cos(t)-\sin(t) \end{bmatrix}\vec{x}_0$$

This took about 10 minutes.

Note: $\vec{w} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ gives $P = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$ takes more work.

2. Find the general solution to $\dot{\vec{x}} = A\vec{x}$, with

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

clearly $\lambda_1 = -2$ with $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenpair.

λ_2 and λ_3 are eigenvalues of $\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$. $\tau = 2$, $\frac{\tau}{2} = 1$, $\delta = 0 - (-2) = 2$

$\lambda_{2,3} = 1 \pm \sqrt{1-2} = 1 \pm i$

let $\lambda_2 = 1+i$, $\vec{w}_2 = \vec{u}_2 + i\vec{v}_2 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in \mathbb{C}^3$.

$(A - \lambda_2 I)\vec{w}_2 = \vec{0}$ is $\begin{bmatrix} -1-i & -2 & 0 \\ 1 & 1-i & 0 \\ 0 & 0 & -3-i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

linearly dependent equations ↪

$(-1-i)\alpha - 2\beta = 0$

$\alpha + (1-i)\beta = 0 \quad \therefore \alpha = (-1+i)\beta$

$(-3-i)\gamma = 0 \quad \therefore \gamma = 0$

Set $\beta = 1$, $\alpha = -1+i$

another choice is $(-1-i)\alpha = 2\beta$, $\alpha = 2$, $\beta = -1-i$; $\alpha = -2$, $\beta = 1+i$

$\vec{w}_2 = \begin{bmatrix} -2 \\ 1+i \\ 0 \end{bmatrix}$

$$\vec{w}_2 = \begin{bmatrix} -1+i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\vec{u}_2$ $\quad$ $\vec{v}_2$

The general solution is $\vec{x}(t) = C_1 e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} +$ continued

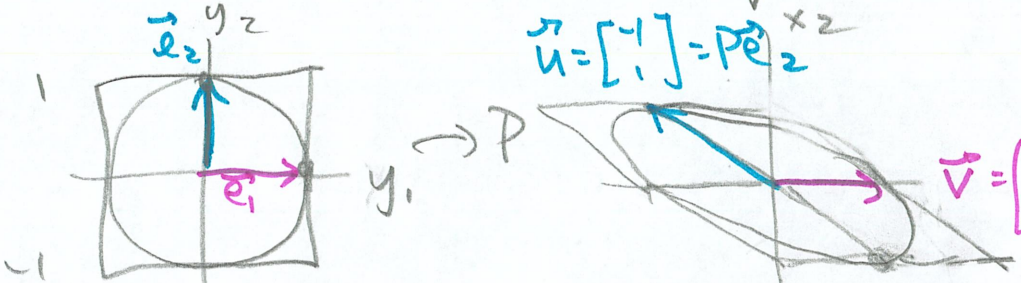
$$\boxed{\vec{x}(t) = C_1 e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + r_0 e^{t}\left(\cos(t+\theta_0)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \sin(t+\theta_0)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)}$$

Note: using the Swift method on the x_1-x_2 invariant subspace, $A_{2\times 2} - 1I_2 = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$, we see that

$\vec{v}_2 = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ gives $\vec{u}_2 = \begin{bmatrix} -a-2b \\ a+b \\ 0 \end{bmatrix}$, so $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ gives $\vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Sketch solution in the x_1-x_2 plane:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = r_0 e^t \left(\cos(t) \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{v}} + \sin(t) \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{u}} \right)$$

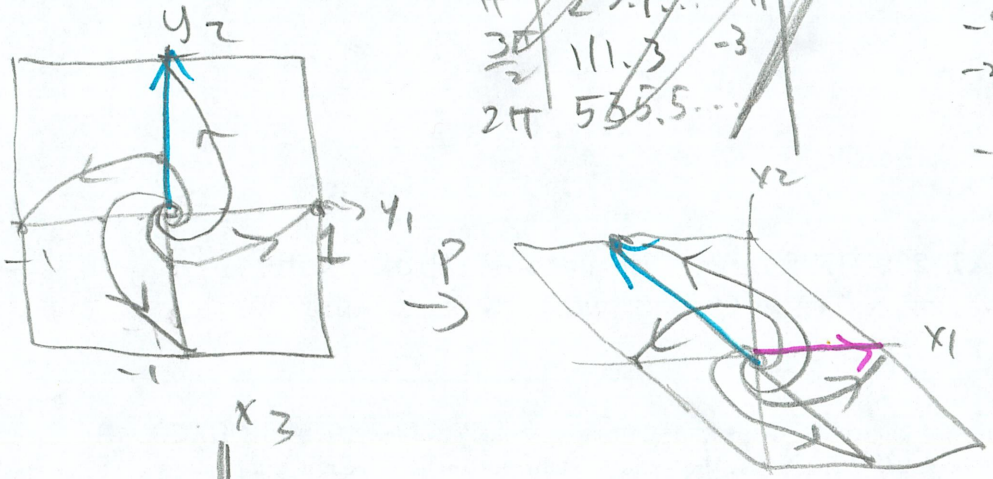


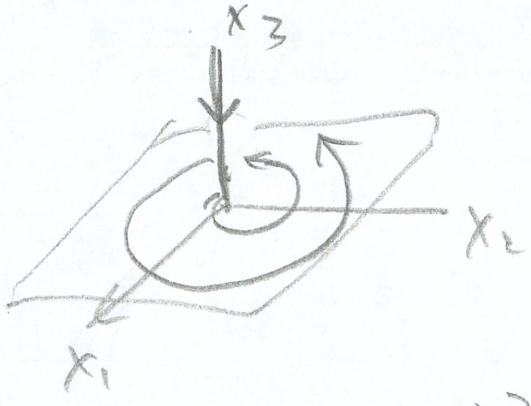
solution in $\vec{y}$

$r = e^\theta$

θ	r
0	1
$\frac{\pi}{2}$	4.81...
π	23.1...
$\frac{3\pi}{2}$	111.3
2π	535.5...

θ	r
0	1
$\frac{\pi}{2}$	0.21...
$-\pi$	0.04...
$-\frac{3\pi}{2}$	0.009...
-2π	0.002...





Use general solution on previous page to determine E_s and E_u.

$E_s = \text{span}\left(\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$ since an I.C. with $r_0 = 0$ gives $\vec{x}(t) \to 0$ as $t \to \infty$

$E_u = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

Since an I.C. with $c_1 = 0$ satisfies $\vec{x}(t) \to \vec{0}$ as $t \to -\infty$.

3. Find the general solution to $\vec{x}' = A\vec{x}$ (5)

with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 1 & 3 & 2 \end{bmatrix}$

$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & -3 \\ 1 & 3 & 2-\lambda \end{bmatrix} = (1-\lambda)\left[(2-\lambda)^2+9\right] = 0$

$-0 \quad +0$

$(1-\lambda)\left((2-\lambda)^2+9\right)=0$

$\lambda = 1$ or $(2-\lambda)^2 = -9$

$2-\lambda = \pm 3i$

$\lambda = 2 \pm 3i$

$\lambda = 1$, $\vec{V}$ satisfies $(A-I)\vec{V} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -3 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0 = 0$

$\beta - 3\gamma = 0 \quad \therefore \beta = 3\gamma$

$\alpha + 3\beta + \gamma = 0 \longrightarrow \alpha + 3(3\gamma) + \gamma = 0$

$\alpha + 10\gamma = 0$

$\gamma = 1, \alpha = -10$

$\beta = 3$

$\vec{V} = \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}$

$\lambda = 2+3i$

$(A-(2+3i)I)\vec{W} = 0$

$$\begin{bmatrix} -2-3i & 0 & 0 \\ 0 & -3i & -3 \\ 1 & 3 & -3i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(-2-3i)\alpha = 0 \quad \therefore \alpha = 0$

$-3i\beta - 3\gamma = 0 \quad \gamma = -i\beta$

$\alpha + 3\beta - 3i\gamma = 0 \quad \gamma = 1, \beta = i$ works

$\vec{W} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ gensdn:

$$\vec{X}(t) = C_1 e^{t}\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix} + r_0 e^{2t}\left(\cos(3t+\theta_0)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin(3t+\theta_0)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

4. Find the general solution to $\vec{x}' = A\vec{x}$, where

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \left[\begin{array}{cc|cc} A_1 & & 0 & \\ \hline 0 & & A_2 & \end{array}\right], \text{ where } A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$$

The eigenvalues of A_1 are...

$\tau = -2$, $\delta = 1+1 = 2$

$\frac{\tau}{2} = -1$ $\quad \lambda = -1 \pm \sqrt{(-1)^2 - 2} = -1 \pm \sqrt{-1} = -1 \pm i$

$a = -1, b = 1$

choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\vec{u}_1 = \frac{1}{1}(A + 1I)\vec{v}_1$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

gen solution is $\vec{x}_1(t) = r_{0,1} e^{-t}\left(\cos(t+\theta_{0,1})\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(t+\theta_{0,1})\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

for $A_2 = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$

$\tau = 2$, $\delta = 0+2$ $\quad \lambda = 1 \pm i$

$\frac{\tau}{2} = 1$

choose $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{u}_2 = \frac{1}{1}(A - I)\vec{v}_2 = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

so $\vec{x}_2 = r_{0,2} e^{t}\left(\cos(t+\theta_{0,2})\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(t+\theta_{0,2})\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

The general solution to the 4-D problem is $\vec{x}(t) = \begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{bmatrix}$

$$\boxed{\vec{x}(t) = r_{0,1} e^{-t}\left(\cos(t+\theta_{0,1})\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sin(t+\theta_{0,1})\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) + r_{2,0} e^{t}\left(\cos(t+\theta_{2,0})\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \sin(t+\theta_{2,0})\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)}$$