Assignment 6 MAT 665, Swift

H1. Sketch the phase portrait of $\vec{\dot{x}} = A\vec{x}$, starting with the phase portrait in the $\vec{y}$-coordinates

$A = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix}$. $\tau = 0$, $\delta = -4 + 20 = 16$, $\lambda = \pm 4i$

$(A - 4iI)\vec{w} = \vec{0}$ $\quad \begin{bmatrix} 2-4i & 4 \\ -5 & -2-4i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let $\vec{w} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$(2-4i)\alpha + 4\beta = 0$

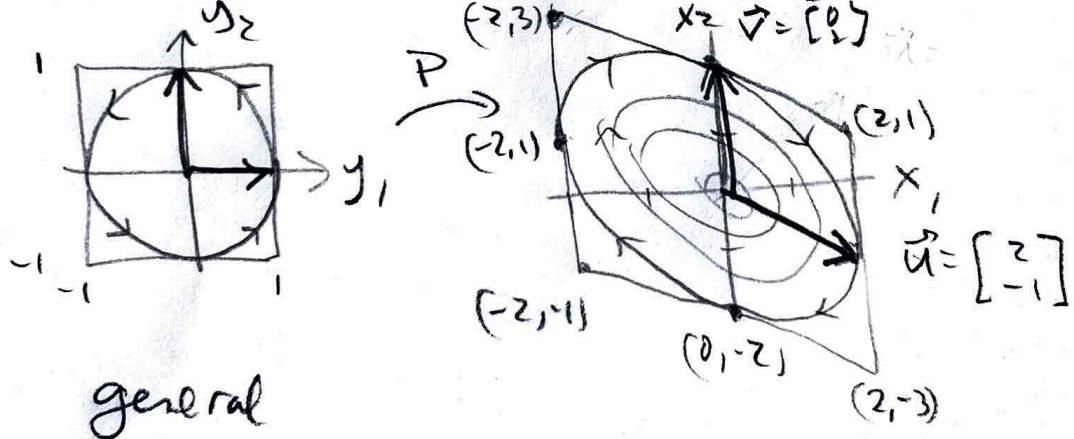
Don't worry about the 2nd equation.

$4\beta = (-2+4i)\alpha$

$\beta = \frac{-1+2i}{2}\alpha$. Let $\alpha = 2$, $\beta = -1+2i$

So $\vec{w} = \begin{bmatrix} 2 \\ -1+2i \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, so $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}$




General Solution is $\vec{x}(t) = r_0\left(\cos(4t+\theta_0)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin(4t+\theta_0)\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$

Sketch one trajectory, the ellipse that touches the centers of the edges of the parallelogram. Then shrink or stretch about the origin (i.e., vary $r_0$) to get several other trajectories.

H2. Same question, with $A = \begin{bmatrix} 1 & 4 \\ -5 & -3 \end{bmatrix}$

Note that $\underbrace{\begin{bmatrix} 1 & 4 \\ -5 & -3 \end{bmatrix}}_{\text{New } A} = \underbrace{\begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix}}_{\text{old } A} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

So the eigenvalues of the new $A$ are $\lambda = -1 \pm 4i$,
and the eigenvectors are the same: $\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

The trajectories in the $\vec{y}$ coordinates are logarithmic spirals, $r = r_0 e^{-\frac{1}{4}(\theta - \theta_0)}$.

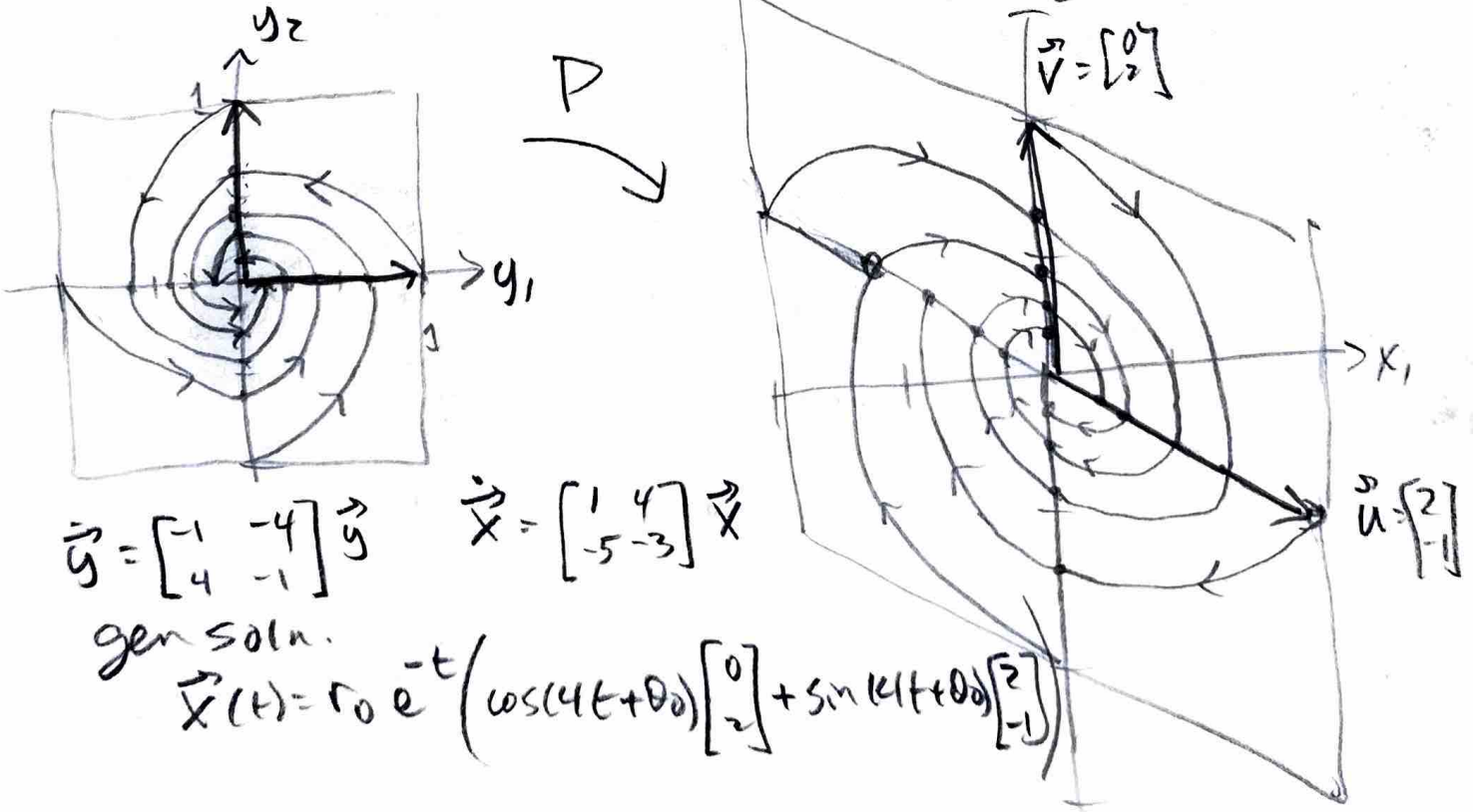
I choose to draw 4 trajectories: $r_0 = 1$, and $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

Quarter turn: $\Delta\theta = \frac{\pi}{2}$, gives $e^{-\frac{\pi}{8}} \approx 0.68$ compression factor

half turn $\Delta\theta = \pi$, gives $e^{-\frac{\pi}{4}} \approx 0.46$ "

3 quarter turn $\Delta\theta = \frac{3\pi}{2}$, gives $e^{-\frac{3\pi}{8}} \approx 0.31$ "

full turn $\Delta\theta = 2\pi$ gives $e^{-\frac{\pi}{2}} \approx 0.21$ "




$\vec{y} = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix} \vec{y}$ $\quad \vec{x} = \begin{bmatrix} 1 & 4 \\ -5 & -3 \end{bmatrix} \vec{x}$

gen soln.

$$\vec{x}(t) = r_0 e^{-t}\left(\cos(4t + \theta_0)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin(4t + \theta_0)\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$$