MAT 665, Differential Equations, Prof. Swift, JCF example

The eigenvalues of the 4×4 matrix A are $2, 2, 2, 2$. Furthermore,

$$A - 2I = \begin{bmatrix} -2 & -1 & -1 & -1 \\ 7 & 1 & 0 & 3 \\ -2 & -1 & -1 & -1 \\ 4 & 2 & 2 & 2 \end{bmatrix}, \quad \text{rref}(A - 2I) = \begin{bmatrix} 1 & 0 & -1/5 & 2/5 \\ 0 & 1 & 7/5 & 1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$(A - 2I)^2 = \begin{bmatrix} -5 & 0 & 1 & -2 \\ 5 & 0 & -1 & 2 \\ -5 & 0 & 1 & -2 \\ 10 & 0 & -2 & 4 \end{bmatrix}, \quad \text{rref}((A - 2I)^2) = \begin{bmatrix} 1 & 0 & -1/5 & 2/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $(A - 2I)^3 = 0$.

(a) Find the Jordan Canonical Form of A. Call it B.

Let $\mathbf{x} = P\mathbf{y}$ and find a matrix P such that $P^{-1}AP = B$ as follows:

(b) Find the equations that the standard basis vectors satisfy in the $\mathbf{y}$ coordinates. Hint. The first one is $(B - 2I)\mathbf{e}_1 = \mathbf{0}$

(c) Let $P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$. Find the equations that the vectors $\mathbf{v}_i = P\mathbf{e}_i$ satisfy. What are the conditions that ensure that P is nonsingular?
Hint: The first equation is $(A - 2I)\mathbf{v}_1 = \mathbf{0}$. The first condition is $\mathbf{v}_1 \neq \mathbf{0}$.

(d) Solve those equations to find P, starting at the top of each tower of generalized eigenvectors. (This is not the algorithm in the book!) Many solutions are possible. I suggest you choose P to have integer elements with small absolute values.

(e) Write down the general solution to the ODE $\dot{\mathbf{y}} = B\mathbf{y}$, using $\mathbf{c} \in \mathbb{R}^4$ as the 4 constants. Write your solution as a single column vector.

(f) Write down the general solution to the ODE $\dot{\mathbf{x}} = A\mathbf{x}$ using the answer to part (e), without doing any more work. In particular, do not evaluate P^{-1}.