Do Set 7, prob 16 first: $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ $\lambda = 2, 2$

$A - 2I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ $(A-2I)^2 = 0$

MAT 665, Differential Equations, Prof. Swift, JCF example

The eigenvalues of the 4×4 matrix A are $2, 2, 2, 2$. Furthermore,

$$A - 2I = \begin{bmatrix} -2 & -1 & -1 & -1 \\ 7 & 1 & 0 & 3 \\ -2 & -1 & -1 & -1 \\ 4 & 2 & 2 & 2 \end{bmatrix}, \quad \text{rref}(A - 2I) = \begin{bmatrix} 1 & 0 & -1/5 & 2/5 \\ 0 & 1 & 7/5 & 1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$\delta_1 = 1$, $\delta_2 = 2$

1 tower of size 2

$$(A - 2I)^2 = \begin{bmatrix} -5 & 0 & 1 & -2 \\ 5 & 0 & -1 & 2 \\ -5 & 0 & 1 & -2 \\ 10 & 0 & -2 & 4 \end{bmatrix}, \quad \text{rref}((A - 2I)^2) = \begin{bmatrix} 1 & 0 & -1/5 & 2/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $(A - 2I)^3 = 0$.

$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

(a) Find the Jordan Canonical Form of A. Call it B.

Let $\mathbf{x} = P\mathbf{y}$ and find a matrix P such that $P^{-1}AP = B$ as follows:

(b) Find the equations that the standard basis vectors satisfy in the $\mathbf{y}$ coordinates. Hint. The first one is $(B - 2I)\mathbf{e}_1 = \mathbf{0}$

(c) Let $P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$. Find the equations that the vectors $\mathbf{v}_i \in \mathbb{R}^4$ satisfy. What are the conditions that ensure that P is nonsingular?
Hint: The first equation is $(A - 2I)\mathbf{v}_1 = \mathbf{0}$. The first condition is $\mathbf{v}_1 \neq \mathbf{0}$.

(d) Solve those equations to find P, starting at the top of each tower of generalized eigenvectors. (This is not the algorithm in the book!) Many solutions are possible. Choose P to have integer elements.

$\vec{y}(t) = e^{2t} \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & t & t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) Write down (by inspection) the general solution to the ODE $\dot{\mathbf{y}} = B\mathbf{y}$, using $\mathbf{c} \in \mathbb{R}^4$ as the 4 constants. Write the solution as a linear combination of $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{e}_4$.

(f) Write down the general solution to the ODE $\dot{\mathbf{x}} = A\mathbf{x}$ using those same constants, without doing any more work. In particular, do not evaluate P^{-1}. as a linear combination of $\{\vec{v}_i\}$.

(a) Deficiency indices $\delta_1 = 2$, $\delta_2 = 3$, $\delta_3 = 4$ = multiplicity

2 towers ($\delta_1 = 2$) and largest tower is 3 tall ($\delta_3 = 4$).

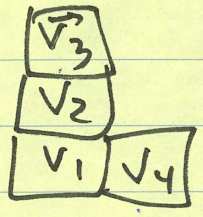
So $B = \text{diag}(J_3(2), J_1(2)) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(b) $(B-2I)\vec{e}_1 = \vec{0}$, $(B-2I)\vec{e}_2 = \vec{e}_1$, $(B-2I)\vec{e}_3 = \vec{e}_2$, $(B-2I)\vec{e}_4 = \vec{0}$.

$(B-2I)^2\vec{e}_3 = \vec{e}_1$

(c) $(A-2I)\vec{v}_1 = \vec{0}$, $(A-2I)\vec{v}_2 = \vec{v}_1$, $(A-2I)\vec{v}_3 = \vec{v}_2$, $(A-2I)\vec{v}_4 = \vec{0}$.

$\vec{v}_1 \neq \vec{0}$, $\vec{v}_1$ and $\vec{v}_4$ are linearly independent ($\Rightarrow \vec{v}_4 \neq \vec{0}$)



$\vec{v}_3$ is in ~~the~~ $\ker(A-2I)^3$ but NOT in $\ker(A-2I)^2$.

~~Any~~ Any $\vec{v} \in \mathbb{R}^4$ satisfies this!

Simplest choice is $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ Note: $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is NOT allowed!

then $\vec{v}_2 = \begin{bmatrix} -2 \\ 7 \\ -2 \\ 4 \end{bmatrix}$ and $\vec{v}_1 = \begin{bmatrix} -5 \\ 5 \\ -5 \\ 10 \end{bmatrix}$

$\vec{v}_4$ is in ~~the~~ $\ker(A-2I)$: $\operatorname{rref}(A-2I) = \begin{bmatrix} 1 & 0 & -\frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{7}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

I choose $\vec{v}_4 = \begin{bmatrix} * \\ * \\ 0 \\ 5 \end{bmatrix}$, so

$\vec{v}_4 = \begin{bmatrix} x \\ y \\ 0 \\ 5 \end{bmatrix}$ so $x - \frac{1}{5}\cdot 0 + \frac{2}{5}\cdot 5 = 0$

$x + 2 = 0 \quad x = -2$

$y + \frac{7}{5}\cdot 5 + \frac{1}{5}\cdot 5 = 0 \quad y = -1$

$\vec{v}_4 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 5 \end{bmatrix}$

~~If~~ ~~Better rule~~ Smaller entries like $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$,

$\vec{v}_1 = \begin{bmatrix} -5 \\ 5 \\ -5 \\ 10 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix}$ choose $\vec{v}_4$ as before.

$P = \begin{bmatrix} 0 & -1 & -2 & -2 \\ 0 & 0 & -2 & 1 \\ 1 & -1 & 2 & 0 \\ 0 & 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix}$

$$\vec{y}(t) = e^{2t}\begin{bmatrix} 1 & t & \frac{t^2}{2} & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\vec{C}$$

$$y(t) = e^{t}\begin{bmatrix} c_1 + tc_2 + \frac{t^2}{2}c_3 \\ c_2 + tc_3 \\ c_3 \\ c_4 \end{bmatrix} \qquad \vec{x}(t) = [v_1 v_2 v_3 v_4]\,y(t)$$

$$\text{So } \vec{x}(t) = e^{t}\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}\begin{bmatrix} c_1 + t(c_2 + \frac{t^2}{2}c_3 \\ c_2 + tc_3 \\ c_3 \\ c_4 \end{bmatrix}$$

$$= e^{t}\left(\vec{v}_1\left(e\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}\left(c_1 + t(c_2 + \frac{t^2}{2}c_3\right) + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}(c_2 + tc_3)\right.\right.$$

$$\left. + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}c_3 + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 5 \end{bmatrix}c_4\right)$$