MAT 665, Differential Equations, Prof. Swift

Exam 1, 2026-03-06, on Sections 1.1-1.6

Name: key.

No book, no notes. Electronic devices, including cell phones and graphing calculators, are not allowed.

You may bring one page of notes, with writing on both sides.

Unless I warn you otherwise, you need not deal with fractions or irrational numbers.

You do you need to invert matrices in this exam. In fact, computing the inverse of any matrix might result in a small reduction of points, since it indicates that you are not a true disciple of mine.

Ask me if you have any questions!

optional things are put in a bubble

1. Find the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$, and sketch the phase portrait fairly accurately, where A is the given matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

$\tau = -2$, $\delta = -3$

$\lambda_1 + \lambda_2 = -2$, $\lambda_1 \cdot \lambda_2 = -3$

$\therefore \lambda_1 = 1$, $\lambda_2 = -3$

or $\lambda = \frac{\tau}{2} \pm \sqrt{\left(\frac{\tau}{2}\right)^2 - \delta}$

$\lambda = -1 \pm \sqrt{-(-3)} = -1 \pm \sqrt{4} = -1 \pm 2$; $\lambda_1 = 1$, $\lambda_2 = -3$

or $\det(A - \lambda I) = 0$,

$\lambda^2 + 2\lambda - 3 = 0$

$(\lambda - 1)(\lambda + 3) = 0$

$\lambda_1 = 1$, $\lambda_2 = -3$

Compute eigenvectors

$\lambda_1 = 1$. $A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{cases} -a + b = 0 \\ 3a - 3b = 0 \end{cases} \therefore a = b$ for $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$

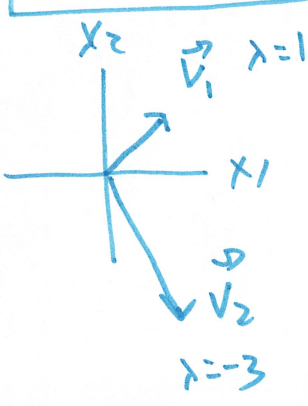
$\lambda_2 = -3$. $A - \lambda_2 I = A + 3I = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$, so $\vec{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

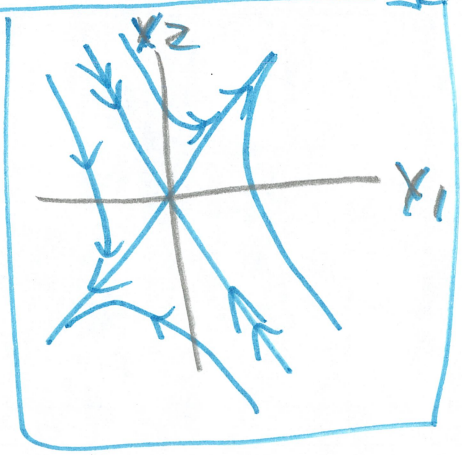
$3a + b = 0$

$b = -3a$

So, the general solution is

$$\vec{x}(t) = C_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$





Note: since $|\lambda_1| < |\lambda_2|$, that is, $1 < 3$, the trajectories get squashed strongly towards the span($\vec{v}_1$) unstable eigenspace

2. Assume that A is a 2×2 real matrix with eigenvalues $\pm i$. Furthermore, an eigenvector for i is $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.

$\lambda = 0 \pm 1i \quad a=0, b=1$

Write down the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$ using the "r_0, θ_0" form, and sketch the phase portrait in the $\mathbf{x}$ plane fairly accurately.

Note: $e^{at} = e^0 = 1$.

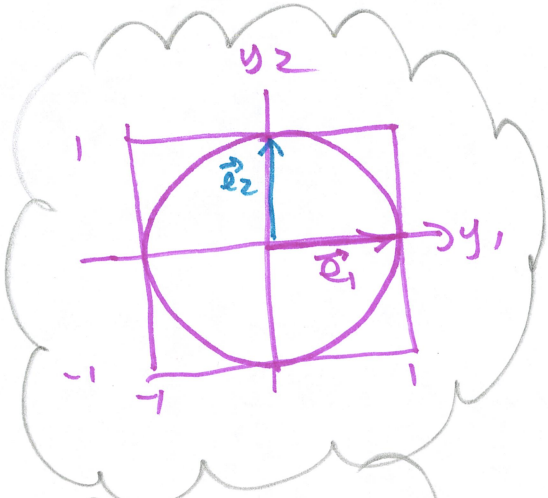
$\vec{w} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

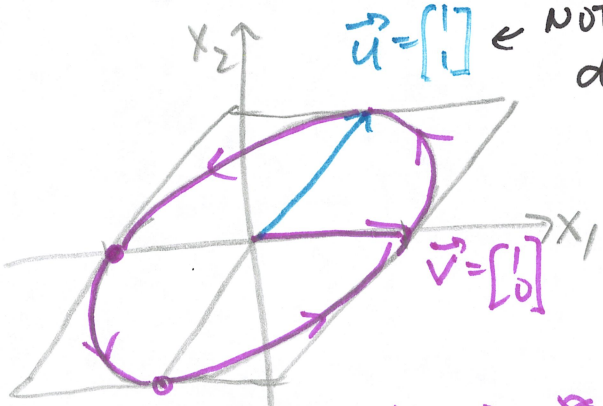
The general solution is

$\vec{x}(t) = r_0 \left(\cos(t+\theta_0) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(t+\theta_0) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

using the formula $\vec{x}(t) = r_0 e^{at} \left(\cos(bt+\theta_0)\vec{v} + \sin(bt+\theta_0)\vec{u} \right)$

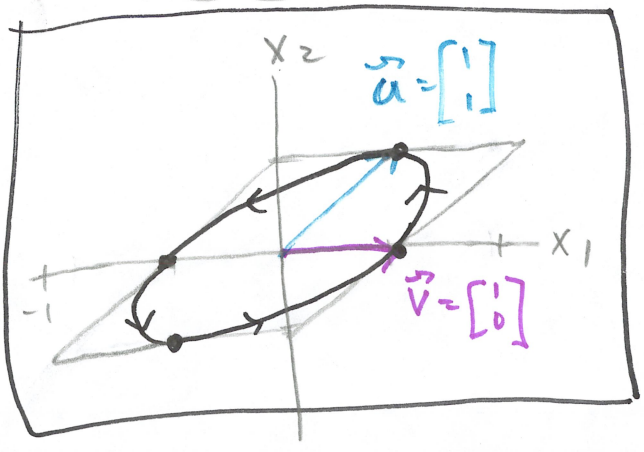


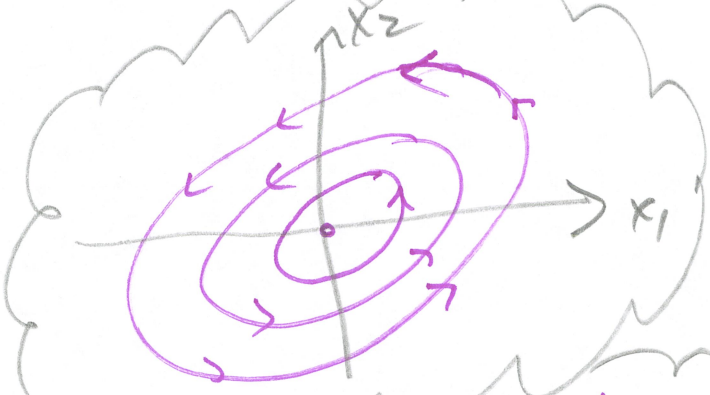
$\vec{y} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{y}$



← NOT accurately drawn!

Trajectory with $r_0 = 1$.





The phase portrait has more than one trajectory, with various values of r_0

3. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

A is upper triangular

(a) Find the eigenvalues and eigenvectors of A. $\lambda_1 = 1,\ \lambda_2 = -2,\ \lambda_3 = 3$

(b) Find the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$.

(c) Find a matrix P, and a diagonal matrix B, such that $P^{-1}AP = B$.

(d) Sketch the 3D phase portrait in $\mathbf{y}$ space, where $\mathbf{x} = P\mathbf{y}$.

$\lambda_1 = 1$, $A - \lambda I = A - I = \begin{bmatrix} 0 & 1 & -2 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ by inspection

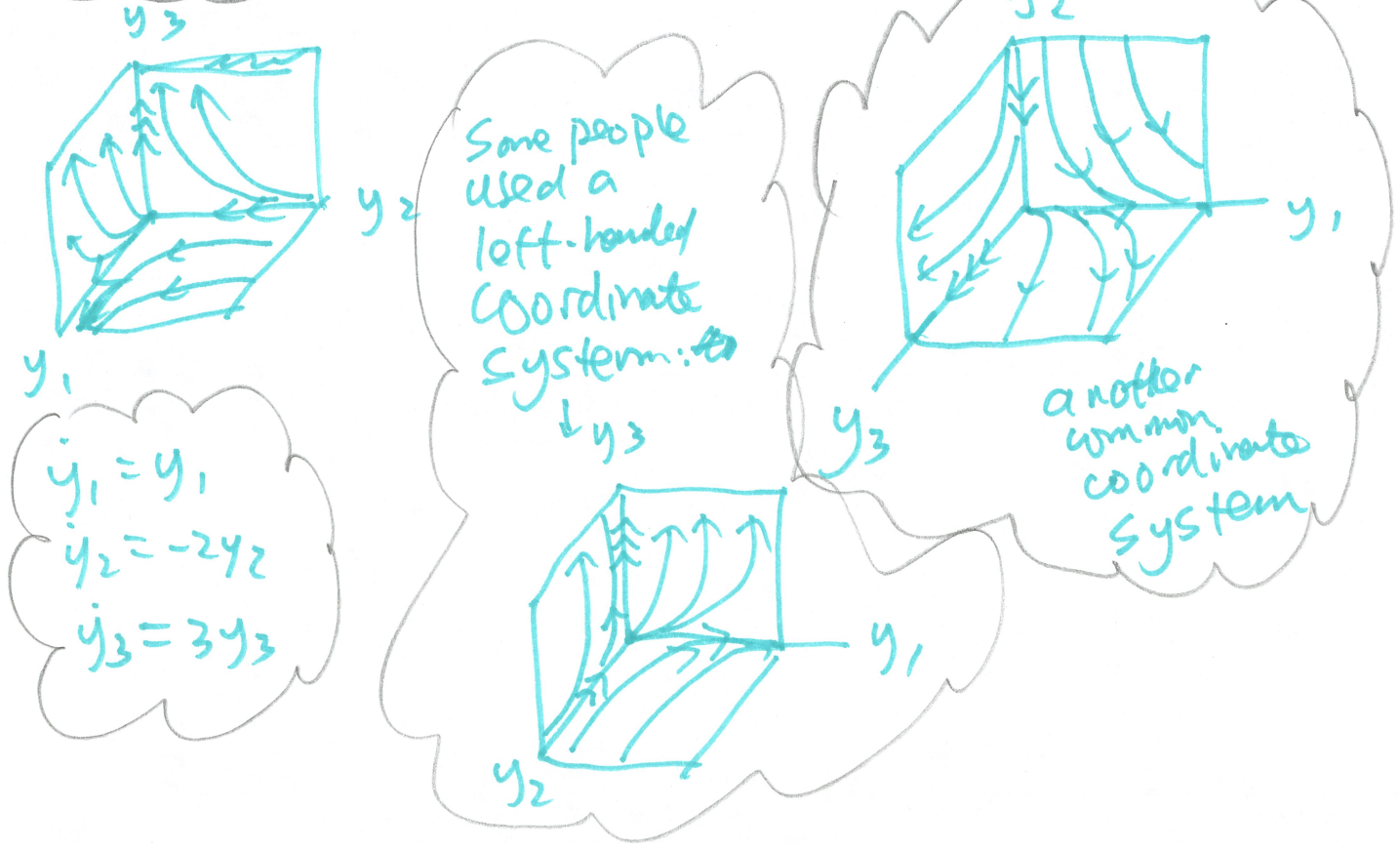
$\lambda_2 = -2$ $A - \lambda_2 I = A + 2I = \begin{bmatrix} 3 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. $\vec{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ by inspection

$\lambda_3 = 3$ $A - \lambda_3 I = A - 3I = \begin{bmatrix} -2 & 1 & -2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ by inspection

So the general solution is $\vec{x}(t) = C_1 e^{t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{-2t}\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + C_3 e^{3t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ yields $P^{-1}AP = B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$P = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$



$\dot{y}_1 = y_1$

$\dot{y}_2 = -2y_2$

$\dot{y}_3 = 3y_3$

In case you need more room, I have repeated the matrix from problem 3 here, and given you more room to do the computations.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Extra Calculations, in case the "by inspection" results were too fast.

Eigenvalues: $\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & -2 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0$ (expand down 1st column)

$(1-\lambda)(-2-\lambda)(3-\lambda) = 0$

So $\lambda_1 = 1$, or $\lambda_2 = -2$, or $\lambda_3 = 3$.

Eigenvectors

$\lambda_1 = 1$: $\vec{v}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $(A+I)\vec{v}_1 = 0$

$\begin{bmatrix} 0 & 1 & -2 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad b-2c=0,\ -3b=0,\ 2c=0$

$b=c=0$, a is free

I choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$;

$\lambda_2 = -2$ $\vec{v}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$; $\begin{bmatrix} 3 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \therefore c=0$

$3a+b=0$, $b=-3a$

choose $a=1$, so $b=-3$

$\vec{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

$\lambda_3 = 3$, $\vec{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\begin{bmatrix} -2 & 1 & -2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \therefore b=0$

$-2a-2c=0$

$c=-a$

choose $a=1$, so $c=-1$

$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(use $\vec{x}(t) = e^{at}(\cos(bt)\vec{v} + \sin(bt)\vec{u})$)

4. The solution to this problem *does* involve fractions and irrational numbers! Consider the ODE

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} \mathbf{x}$$

(a) Use the "Swift method" to solve the ODE with the initial condition $\mathbf{x}(0) = (1, -1)$. $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) Use the "newly discovered Swift method" to evaluate $\exp(At)$. As a hint, recall that the ODE with the arbitrary initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x}(t) = \exp(At)\mathbf{x}_0$.

$\tau = 2$, $\delta = 0 - (-3) = 3$, so $\lambda = 1 \pm \sqrt{1^2 - 3}$

$\frac{\tau}{2} = 1$

$= 1 \pm \sqrt{-2}$

$\boxed{\lambda = 1 \pm \sqrt{2}\, i}$

$a = 1, \quad b = \sqrt{2}$

Many people did (b) before (a). That is more work.

(a). choose $\vec{w} = \vec{u} + i\vec{v}$ with $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

then $\vec{u} = \frac{1}{b}(A - aI)\vec{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ Now use this

The solution with $\vec{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is

$$\boxed{\vec{x}(t) = e^t \left(\cos(\sqrt{2}t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{\sin(\sqrt{2}t)}{\sqrt{2}} \begin{bmatrix} -2 \\ -4 \end{bmatrix} \right)}$$

← I prefer this form.

Or, this can be written as

$$\boxed{\vec{x}(t) = e^t \begin{bmatrix} \cos(\sqrt{2}t) - \frac{2}{\sqrt{2}} \sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \frac{4}{\sqrt{2}} \sin(\sqrt{2}t) \end{bmatrix}}$$

(b) Let $\vec{v} = \vec{x}_0$, then $\vec{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -3 & 1 \end{bmatrix} \vec{x}_0$, and

$$\vec{x}(t) = e^t \left(\cos(\sqrt{2}t)\vec{x}_0 + \frac{\sin(\sqrt{2}t)}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -3 & 1 \end{bmatrix} \vec{x}_0 \right)$$ Note. $\vec{x}_0 = I_2 \vec{x}_0$.

So

$$\boxed{e^{At} = e^t \begin{bmatrix} \cos(\sqrt{2}t) - \frac{\sin(\sqrt{2}t)}{\sqrt{2}} & \frac{\sin(\sqrt{2}t)}{\sqrt{2}} \\ -3\frac{\sin(\sqrt{2}t)}{\sqrt{2}} & \cos(\sqrt{2}t) + \frac{\sin(\sqrt{2}t)}{\sqrt{2}} \end{bmatrix}}$$