MAT665
Perko §1.7 #26 (by hand)

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda = -1, -1, 1$$

generators with eigenvalue -1:

$$A+I = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix} \qquad (A+I)^2 = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

2 generalized eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

eigenvector for 1 (call it $\vec{v}_3$)

$$A - I = \begin{bmatrix} -2 & 1 & -2 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{let } \vec{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (A-I)\vec{v}_3 = \begin{bmatrix} a \\ b-2c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So we have $a=0$, $b-2c=0$ choose $c=1$, so $b=2$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad P = [\vec{v}_1 \vec{v}_2 \vec{v}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Find } P^{-1} \text{ by}$$

row reduction

$$[P \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \therefore P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

The semisimple part of (a) is

$$S = P \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} P^{-1} = P \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

check if $NS = SN$

$$NS = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$SN = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ yes!

Note that $N^2 = 0$, so $e^{Nt} = I + Nt = \begin{bmatrix} 1 & t & -2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$e^{St} = P\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}P^{-1} = P\begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{t} & -2e^{t} \\ 0 & 0 & e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & -2e^{-t}+2e^{t} \\ 0 & 0 & e^{t} \end{bmatrix}$$

$$e^{At} = e^{St}e^{Nt} = \begin{bmatrix} e^{-t} & te^{-t} & -2te^{-t} \\ 0 & e^{-t} & -2e^{-t}+2e^{t} \\ 0 & 0 & e^{t} \end{bmatrix}$$

solution to (1) is $\vec{x}(t) = e^{At}\vec{x}_0$

Note: $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

semisimple ↖ ↗ Nilpotent

But those two matrices don't commute.

Don't fall into that trap. The true $S+N$ decomposition, $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, is not obvious.