COMPLEX EIGENVALUES

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This handout concerns the solution to $\dot{\mathbf{x}} = A\mathbf{x}$, where A is a square matrix with real entries and simple complex (non-real) eigenvalues.

For simplicity, assume that A is 2×2. We will do other cases by example. Following Perko's book, assume that $\mathbf{A}$ has eigenvalues $\lambda = a \pm ib$, with $a, b \in \mathbb{R}$ and $b > 0$. The eigenvector $\mathbf{w} \in \mathbb{C}^2$ for $\lambda = a + ib$ satisfies

$$A\mathbf{w} = (a + ib)\mathbf{w},$$

Since A has real entries, $A\overline{\mathbf{w}} = (a - ib)\overline{\mathbf{w}}$. We can write the eigenvector $\mathbf{w}$ in terms of its real and imaginary parts as

$$\mathbf{w} = \mathbf{u} + i\mathbf{v}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. There is quite a bit of freedom in choosing $\mathbf{u}$ and $\mathbf{v}$, since $\alpha\mathbf{w}$ is an eigenvector of A for any $\alpha \in \mathbb{C}$.

We can write a (complex-valued) solution to $\dot{\mathbf{x}} = A\mathbf{x}$ as $\mathbf{x}(t) = e^{(a+ib)t}(\mathbf{u} + i\mathbf{v})$. Using Euler's formula for e^{ibt} and skipping some work, we find

$$\mathbf{x}(t) = e^{at}\left(\cos(bt)\mathbf{u} - \sin(bt)\mathbf{v}\right) + ie^{at}\left(\cos(bt)\mathbf{v} + \sin(bt)\mathbf{u}\right).$$

The real and imaginary parts of the previous expression are both real-valued functions. Thus the general real-valued solution can be written as

$$\mathbf{x}(t) = c_1 e^{at}\left(\cos(bt)\mathbf{v} + \sin(bt)\mathbf{u}\right) + c_2 e^{at}\left(\cos(bt)\mathbf{u} - \sin(bt)\mathbf{v}\right). \tag{1}$$

This is what you find in the undergraduate textbooks. Note that the general solution (1) can be written as

$$\mathbf{x}(t) = e^{at}\begin{bmatrix} \mathbf{v} & \mathbf{u} \end{bmatrix}\begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{2}$$

where the two arbitrary constants are c_1 and c_2. Following Perko, define $P = [\mathbf{v}\ \mathbf{u}]$. The theorem in section 1.4 implies that

$$B := P^{-1}AP = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Let $\mathbf{x} = P\mathbf{y}$ as usual. Here is how to sketch the phase portrait in the $\mathbf{x}$ coordinates: The solutions to

$$\dot{\mathbf{y}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\mathbf{y}$$

are logarithmic spirals (or circles, if $a = 0$) with $r = r_0 \exp\left(\frac{a}{b}(\theta - \theta_0)\right)$. We draw one or more solutions in the $\mathbf{y}$ coordinates along with the square with vertices $\mathbf{y} = (\pm 1, \pm 1)$. Then we draw the vectors $\mathbf{v}$ and $\mathbf{u}$, along with the parallelogram with vertices $\mathbf{x} = \pm\mathbf{v} \pm \mathbf{u}$. The parallelogram is the image of the square under P. We then sketch the image of the logarithmic spirals under P.

Examples: (Done in class) Sketch the phase portraits for $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$, and for $A = \begin{bmatrix} 0 & 5 \\ -2 & -2 \end{bmatrix}$.

If we let $\mathbf{y}(0) = r_0(\cos(\theta_0), \sin(\theta_0))$, then the solution in Equation (2) is

$$\mathbf{y}(t) = r_0 e^{at}\left(\cos(bt + \theta_0)\mathbf{e}_1 + \sin(bt + \theta_0)\mathbf{e}_2\right).$$

Multiply both sides of the solution on the left by P, and note that $P\mathbf{e}_1 = \mathbf{v}$ and $P\mathbf{e}_2 = \mathbf{u}$, to get

$$\mathbf{x}(t) = r_{\mathrm{o}} e^{at}\left(\cos(bt + \theta_{\mathrm{o}})\mathbf{v} + \sin(bt + \theta_{\mathrm{o}})\mathbf{u}\right). \tag{3}$$

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This is useful for drawing the phase portrait. The direction of motion sweeps from $\mathbf{v}$ to $\mathbf{u}$.

As we will see, the form of the general solution in equation (3) can be used to compute the solution to the IVP $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_o$, and to compute $\exp(At)$, very efficiently. However, we return to Equation (2), to reproduce the solution as written in Perko's book. Since $\mathbf{x}(0) = \mathbf{x}_o$, the constants satisfy $P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{x}_o$, which can be solved to give $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}_o$ so the solution to the IVP is

$$\mathbf{x}(t) = e^{at} P \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix} P^{-1}\mathbf{x}_o, \tag{4}$$

which is Perko's solution, using the known result for $\exp(Bt)$.

Example: Suppose that A is a real 2×2 matrix, and $A\mathbf{w} = 2i\mathbf{w}$, where $\mathbf{w} = \begin{bmatrix} 1+4i \\ 3 \end{bmatrix}$. We can write down the general solution by inspection, using equation (3):

$$\mathbf{x}(t) = r_o \left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} \cos(2t + \theta_o) + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \sin(2t + \theta_o) \right),$$

and easily sketch the phase portrait. Note that the P matrix does not need to be inverted, and no matrix multiplication is needed. However, if P^{-1} is computed we can figure out A in this case by using $P = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. Then some matrix algebra shows that

$$A = PBP^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{17}{6} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Do not be alarmed during an exam if I present the information about a matrix in a form similar to the first sentence of this example. I am trying to be nice to you by giving you the eigenvalues and eigenvectors on platter.

Example: For the example on pages 29 and 30 of Perko's book, we can use the eigenvalues and eigenvectors computed in the book and then immediately write down the general solution using Equation (3). Here it is natural to use the constants r_1 and θ_1 for the part of the solution associated with $\lambda_1 = 1 + i$, and r_2 and θ_2 for the part coming from $\lambda_2 = 2 + i$.

$$\mathbf{x}(t) = r_1 e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cos(t + \theta_1) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \sin(t + \theta_1) \right) + r_2 e^{2t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cos(t + \theta_2) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \sin(t + \theta_2) \right).$$

Example: In the example on pages 30 and 31, the real matrix A satisfies $A\mathbf{v}^{(1)} = -3\mathbf{v}^{(1)}$ and $A\mathbf{w}^{(2)} = (2 + i)\mathbf{w}^{(2)}$, where

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w}^{(2)} = \mathbf{u}^{(2)} + i\mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ 1 + i \\ 1 \end{bmatrix}.$$

The general solution in this case is

$$\mathbf{x}(t) = c_1 e^{-3t} \mathbf{v}^{(1)} + r_o e^{2t} \left(\mathbf{v}^{(2)} \cos(t + \theta_o) + \mathbf{u}^{(2)} \sin(t + \theta_o) \right),$$

where I have used the constants r_o and θ_o for the complex eigenvalue. It is clear from this vector form of the general solution that the stable subspace $E^s = \text{span}\{\mathbf{v}^{(1)}\}$ and the unstable subspace $E^u = \text{span}\{\mathbf{u}^{(2)}, \mathbf{v}^{(2)}\}$ are invariant. (That is, an initial condition in one of these invariant subspaces leads to a solution that stays in the invariant subspace for all time.)

1. The Swift Method

Using the form of the solution to $\dot{\mathbf{x}} = A\mathbf{x}$ in Equation (3), we can solve the initial value problem with $\mathbf{x}(0) = \mathbf{x}_0$ very efficiently. The key is to choose an eigenvector of $a + ib$ with $\mathbf{v} = \mathbf{x}_0$. This can always be done (unless $\mathbf{x}_0 = \mathbf{0}$). To see this, assume the eigenvalues of $A \in \mathbb{R}^2$ are $\lambda = a \pm ib$, with $b > 0$. The eigenvalue equation is

$$A(\mathbf{u} + i\mathbf{v}) = (a + ib)(\mathbf{u} + i\mathbf{v}) = a\mathbf{u} - b\mathbf{v} + i(a\mathbf{v} + b\mathbf{u}).$$

The real and imaginary parts of this equation are

$$A\mathbf{u} = a\mathbf{u} - b\mathbf{v}, \quad \text{and} \quad A\mathbf{v} = a\mathbf{v} + b\mathbf{u}. \tag{5}$$

For an arbitrary $\mathbf{v}$, the second equation can be solved for $\mathbf{u}$ as $\mathbf{u} = \frac{1}{b}(A - aI_2)\mathbf{v}$. Let $\mathbf{v} = \mathbf{x}_0$, and choose $r_o = 1$, $\theta_o = 0$ in Equation (3), and the resulting solution to $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{at}\left(\cos(bt)\mathbf{x}_0 + \tfrac{\sin(bt)}{b}(A - aI_2)\mathbf{x}_0\right). \tag{6}$$

Note that this expression is unchanged if b is replaced by $-b$, as it must be, since the eigenvalues of A are $a \pm ib = a \pm i(-b)$.

Since Equation (6) is somewhat complicated, I usually present this to my MAT 239 students in this way: First, compute $\mathbf{x}_1 = (A - aI_2)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{at}\left(\cos(bt)\mathbf{x}_0 + \frac{\sin(bt)}{b}\mathbf{x}_1\right).$$

I am curious to know if other people have discovered Equation (6). I did not know about it the last time I taught MAT 665. In the past week, I have realized that we can go one step farther: If the real 2×2 matrix A has eigenvalues $a \pm ib$, with $a, b \in \mathbb{R}$ and $b \neq 0$, then the solution to $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ for an arbitrary $\mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{at}\left(\cos(bt)I_2 + \tfrac{\sin(bt)}{b}(A - aI_2)\right)\mathbf{x}_0. \text{ Hence } e^{At} = e^{at}\left(\cos(bt)I_2 + \tfrac{\sin(bt)}{b}(A - aI_2)\right). \tag{7}$$

This is a whole lot simpler than computing the matrix exponential as in Equation (4). Furthermore, eigenvalues involving square roots do not make the computation much more complicated.

Example: (Done in class) Let $A = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$. (a) Find the solution to the IVP $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}_0 = (3, 2)$. (b) Evaluate e^{At}.

Additional homework problems for Asignment 6:

H1. Let $A = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix}$. Sketch the phase portrait of $\dot{\mathbf{x}} = A\mathbf{x}$, starting with the phase portrait in the $\mathbf{y}$ coordinates.

H2. Let $A = \begin{bmatrix} 1 & 4 \\ -5 & -3 \end{bmatrix}$. Sketch the phase portrait of $\dot{\mathbf{x}} = A\mathbf{x}$, starting with the phase portrait in the $\mathbf{y}$ coordinates.