MAT 665, Differential Equations, Prof. Swift
Exam 1, 10-27-2017, on Sections 1.1-1.9

No book, no notes. Electronic devices, including cell phones and graphing calculators, are not allowed. You need not deal with irrational numbers, nor do you need to invert matrices, in this exam. In fact, computing the inverse of any matrix will result in a token reduction of points, since it indicates that you are not a true disciple.

Ask me if you have any questions!

Do your work on your own paper. Write your name on each sheet, and indicate what problem you are doing. You may do more than one problem on a sheet, but please use only the front of each page.

1. (a) Find the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$, and sketch the phase portrait fairly accurately, where A is the given matrix:

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}.$$

(b) Write down a fundamental matrix solution, $\Phi(t)$, for $\dot{\mathbf{x}} = A\mathbf{x}$. Recall that $\Phi(t)$ is any nonsingular matrix that satisfies $\Phi'(t) = A\Phi(t)$ for all t.

2. Assume that A is a 2×2 real matrix with eigenvalues $\pm i$. Furthermore, an eigenvector for i is $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$. Find the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$, and sketch the phase portrait.

3. The eigenvalues of the 4×4 matrix A are $2, 2, 2, 2$. Furthermore,

$$A - 2I = \begin{bmatrix} -2 & -1 & -1 & -1 \\ 7 & 1 & 0 & 3 \\ -2 & -1 & -1 & -1 \\ 4 & 2 & 2 & 2 \end{bmatrix}, \quad \text{rref}(A - 2I) = \begin{bmatrix} 1 & 0 & -1/5 & 2/5 \\ 0 & 1 & 7/5 & 1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$(A - 2I)^2 = \begin{bmatrix} -5 & 0 & 1 & -2 \\ 5 & 0 & -1 & 2 \\ -5 & 0 & 1 & -2 \\ 10 & 0 & -2 & 4 \end{bmatrix}, \quad \text{rref}((A - 2I)^2) = \begin{bmatrix} 1 & 0 & -1/5 & 2/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $(A - 2I)^3 = 0$.

(a) Find the Jordan Canonical Form of A. Call it B.

(b) Find a matrix P such that $P^{-1}AP = B$.

4. Consider the matrix

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}.$$

(a) Compute e^{At} using the "$S + N$" method.

(b) Find matrices P and B such that $P^{-1}AP = B$ is in Jordan Canonical Form.

(c) Compute e^{Bt} and use it to find the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$

5. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 1 & -3 & -1 \\ 0 & -2 & -5 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$.

(c) Write down a matrix P such that $P^{-1}AP = B = \text{diag}(1, -2, 3)$.

(d) Sketch the 3D phase portrait in $\mathbf{y}$ space, where $\mathbf{x} = P\mathbf{y}$.