665 MT1 Swift F'2017

1. $A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$ $\quad \tau = -6, \quad \delta = 8 - 3 = 5$

$\lambda = -3 \pm \sqrt{(-3)^2 - 5}$

$= -3 \pm \sqrt{4} = -3 \pm 2$

$\lambda = -5, -1$

$\lambda_1 = -1$

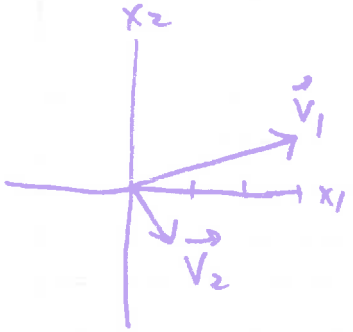
$A - \lambda_1 I = A + I = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$

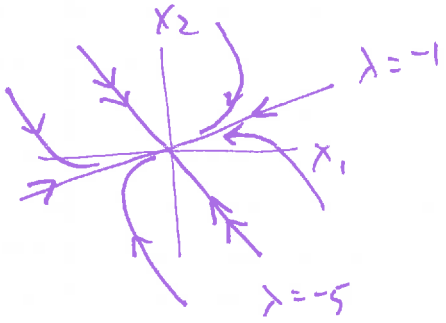
$\vec{V}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda_2 = -5 \quad A - \lambda_2 I = A + 5I = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \quad \vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

general solution is

$$\vec{X}(t) = c_1 e^{-t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$





$$\Phi(t) = \begin{bmatrix} 3e^{-t} & e^{-5t} \\ e^{-t} & -e^{-5t} \end{bmatrix}$$

(each column is a solution.)

I changed problem. See later page.

2. $\lambda_1 = -2$ has $\vec{V}_1 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ $\lambda_+ = 1+3i$ has $\vec{V}_+ = \begin{bmatrix} 2+5i \\ 1 \\ 3 \end{bmatrix}$

$\vec{V}_R = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{V}_I = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{X}(t) = C_1 e^{-2t}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + r_0 e^{t}\left(\cos(\theta_0+3t)\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + \sin(\theta_0+3t)\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right)$$

3. $\delta_1 = 2$, $\delta_2 = 3$, $\delta_3 = 4$, so blocks are 

$\vec{V}_1^{(3)}$
$\vec{V}_1^{(2)}$
$\vec{V}_1^{(1)}$ $\vec{V}_2^{(1)}$ ↓ $A-\lambda I$ moves vectors down.

and $B = \text{diag}(J_3(2), J_1(2)) = \begin{bmatrix} 2 & 1 & 0 & \\ 0 & 2 & 1 & \\ 0 & 0 & 2 & \\ & & & 2 \end{bmatrix}$

Any vector satisfies $(A-2I)^3\vec{v} = 0$. I choose $\vec{V}_1^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

So $\vec{V}_1^{(2)} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{V}_1^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix} \neq \vec{0}$ (other choices are possible, but not $[0\,1\,0\,0]^T$.)

$\vec{V}_2^{(1)}$ is any eigenvector linearly indep. from $\vec{V}_1^{(1)}$.

choose free variables 0 and 5. $\vec{V}_2^{(1)} = \begin{bmatrix} a \\ b \\ 0 \\ 5 \end{bmatrix}$, where

$a - \frac{1}{5}0 + \frac{2}{5}\cdot 5 = 0 \quad a = -2$

$b + \frac{7}{5}0 + \frac{1}{5}\cdot 5 = 0 \quad b = -1$

$\vec{V}_2^{(1)} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 5 \end{bmatrix}$

So $P = \left[\vec{V}_1^{(1)}\ \vec{V}_1^{(2)}\ \vec{V}_1^{(3)}\ \vec{V}_2^{(1)}\right] = \begin{bmatrix} 1 & -1 & 0 & -2 \\ -1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ -2 & 2 & 0 & 5 \end{bmatrix}$

These choices gave the smallest integers. Most people chose $\vec{V}_1^{(3)} = [1\,0\,0\,0]$.

4. $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

$\tau = 6, \delta = 8+1 = 9$

$\lambda = 3 \pm \sqrt{3^2 - 9} = 3, 3.$

$\lambda = 3, 3$, so $S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, and $N = A - 3I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$N^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, as expected

$e^{At} = e^{(S+N)t} = e^{3t}e^{Nt} = e^{3t}(I + Nt)$

$$\boxed{e^{At} = e^{3t}\begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}}$$

$\boxed{\begin{matrix} V_1^{(2)} \\ V_1^{(1)} \end{matrix}}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$\vec{V}_1^{(2)}$ is any vector with $(A-3I)\vec{V}_1^{(2)} \neq \vec{0}$

choose $\vec{V}_1^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $\vec{V}_1^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boxed{P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}$

$e^{Bt} = e^{3t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

general solution is

$\vec{X}(t) = P e^{Bt}\vec{C} = e^{3t}\begin{bmatrix} \vec{V}_1^{(1)} & \vec{V}_1^{(2)} \end{bmatrix}\begin{bmatrix} C_1 + C_2 t \\ C_2 \end{bmatrix} = \begin{pmatrix} \vec{V}_1^{(1)}(C_1 + C_2 t) \\ + V_1^{(2)} C_2 \end{pmatrix} e^{3t}$

$$\boxed{\vec{X}(t) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}(C_1 + C_2 t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} C_2\right) e^{3t}}$$

or $\vec{X}(t) = \underbrace{e^{3t}}\begin{bmatrix} \vec{V}_1^{(1)} & \vec{V}_1^{(2)} \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \left(\vec{V}_1^{(1)} C_1 + \left(\vec{V}_1^{(1)} t + \vec{V}_1^{(2)}\right) C_2\right) e^{3t}$

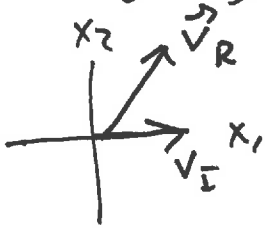
$$\boxed{X(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} C_1 + e^{3t}\begin{bmatrix} 1+t \\ t \end{bmatrix} C_2}$$

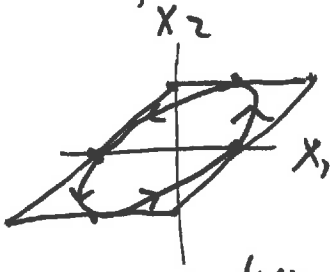
2. $\lambda = \pm i$, $\lambda = i$ has eval $\vec{V} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{V}_R \qquad \vec{V}_I$

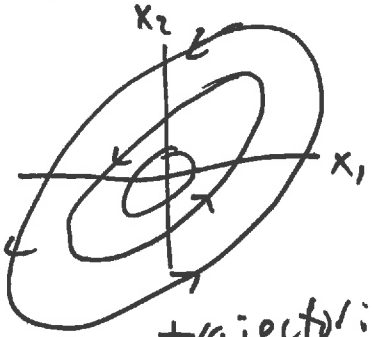
gen soln is $\boxed{\vec{x}(t) = r_0 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t+\theta_0) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(t+\theta_0) \right)}$

Trajectory is an ellipse that sweeps from $\vec{V}_I$ to $\vec{V}_R$.





one trajectory
($r_0 = 1$.)



many trajectories
(various r_0)

5. A is upper triangular, so $\lambda_1 = 1, \lambda_2 = -2, \lambda_3 = 3$

$\vec{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ by inspection

$A - \lambda_2 I = A + 2I = \begin{bmatrix} 3 & -3 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & -3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} +3 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

so $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$A - \lambda_3 I = A - 3I = \begin{bmatrix} -2 & -3 & -1 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

so $\vec{V}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and gen soln. is $\boxed{\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$

$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ gives $\vec{y} = \begin{bmatrix} 1 & -2 \\ & 3 \end{bmatrix} y$



11.