REPEATED COMPLEX EIGENVALUES

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This handout shows how to find the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$, where $A \in \mathbb{R}^{n\times n}$ is a square matrix with real entries and **repeated** complex (non-real) eigenvalues.

For simplicity, assume that A is a 4×4 real matrix with eigenvalues $a \pm ib$, $a \pm ib$ such that $a, b \in \mathbb{R}$ and $b > 0$.

Let $\lambda = a + ib$. To find the Jordan canonical form of A, compute the reduced row echelon form of $A - \lambda I \in \mathbb{C}^4$, probably using symbolic MATLAB. As in the real case, δ_1 is the number of zero rows in $\text{rref}(A - \lambda I)$. If $\delta_1 = 1$, then we want compute $\text{rref}((A - \lambda I)^2)$, which must have 2 zero rows, since $\delta_2 = 2$.

Case 1: $\delta_1 = 1$

In this case B has a 2×2 Jordan block with eigenvalue λ. Since A is real, there must also be one Jordan block with eigenvalue $\overline{\lambda}$, and the Jordan canonical form of A is

$$B = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \overline{\lambda} & 1 \\ 0 & 0 & 0 & \overline{\lambda} \end{bmatrix}.$$

Row reduce $(A - \lambda I)^2$, and find two complex vectors $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ for $j = 1$ and 2 such that

$$(A - \lambda I)^2\mathbf{w}_2 = \mathbf{0} \quad \text{and} \quad (A - \lambda I)\mathbf{w}_2 = \mathbf{w}_1 \neq \mathbf{0}.$$

Note that $\mathbf{w}_1$ is automatically an eigenvector for λ. The matrix $P = [\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \overline{\mathbf{w}_1} \;\; \overline{\mathbf{w}_2}]$ puts A into Jordan canonical form. That is, $P^{-1}AP = B$. My philosophy is that I allow computer algebra to compute the reduced row echelon forms and P^{-1}. We should certainly check that $P^{-1}AP = B$, and if needed, we can compute $\exp(At) = P\exp(Bt)P^{-1}$. However, the result of this computation gives no geometric information about the solution, unlike the following method. As a bonus, this method finds the general solution without computing P^{-1}.

The real-valued general solution to $\dot{\mathbf{x}} = A\mathbf{x}$ can be written $\mathbf{x}(t) = Pe^{Bt}\mathbf{c}$, where the complex constants $\mathbf{c} \in \mathbb{C}^4$ are chosen to make $\mathbf{x}(t)$ real-valued. Let $c_1 = -ir_1e^{i\theta_1}/2$, $c_2 = -ir_2e^{i\theta_2}/2$, $c_3 = \overline{c_1}$ and $c_4 = \overline{c_2}$. With this choice, the general solution with real parameters r_1, r_2, θ_1, and θ_2 is

$$\mathbf{x}(t) = \tfrac{1}{2}\begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{bmatrix} e^{at}e^{ibt} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -ir_1e^{i\theta_1} \\ -ir_2e^{i\theta_2} \end{bmatrix} + \text{c.c.}$$

where "+c.c." means to add the complex conjugate of what came before. Moving the factor of "$-i$" from the last column vector to the front, and using $-i\mathbf{w}_j = \mathbf{v}_j - i\mathbf{u}_j$, we get

$$\mathbf{x}(t) = \tfrac{1}{2}e^{at}\begin{bmatrix}\mathbf{v}_1 - i\mathbf{u}_1 & \mathbf{v}_2 - i\mathbf{u}_2\end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1e^{i(bt+\theta_1)} \\ r_2e^{i(bt+\theta_2)} \end{bmatrix} + \text{c.c.}$$

Using Euler's Formula $e^{i(bt+\theta_j)} = \cos(bt + \theta_j) + i\sin(bt + \theta_j)$, and the fact that

$$\tfrac{1}{2}(\mathbf{v} - i\mathbf{u})(\cos(\alpha) + i\sin(\alpha)) + c.c. = \text{Re}((\mathbf{v} - i\mathbf{u})(\cos(\alpha) + i\sin(\alpha))) = \mathbf{v}\cos(\alpha) + \mathbf{u}\sin(\alpha),$$

the general solution can be written in manifestly real-valued form as

$$\mathbf{x}(t) = e^{at}\left(\begin{array}{l} r_1\Big(\mathbf{v}_1\cos(bt + \theta_1) + \mathbf{u}_1\sin(bt + \theta_1)\Big) + \\ r_2\Big((\mathbf{v}_2 + t\mathbf{v}_1)\cos(bt + \theta_2) + (\mathbf{u}_2 + t\mathbf{u}_1)\sin(bt + \theta_2)\Big) \end{array}\right). \tag{1}$$

Note that $\text{span}(\mathbf{v}_1, \mathbf{u}_1)$ is a 2-dimensional invariant subspace of $\mathbb{R}^4$.

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Case 2: $\delta_1 = 2$

In this case A is diagonalizable over $\mathbb{C}$. In other words, A is semisimple. The Jordan canonical form is

$$B = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \overline{\lambda} & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{bmatrix}.$$

Find two linearly independent eigenvectors for λ; call them $\mathbf{w}_1 = \mathbf{u}_1 + i\mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{u}_2 + i\mathbf{v}_2$. The general real-valued solution to $\dot{\mathbf{x}} = A\mathbf{x}$ can be written

$$\mathbf{x}(t) = e^{at}\left(r_1\left(\mathbf{v}_1\cos(bt+\theta_1) + \mathbf{u}_1\sin(bt+\theta_1)\right) + r_2\left(\mathbf{v}_2\cos(bt+\theta_2) + \mathbf{u}_2\sin(bt+\theta_2)\right)\right). \tag{2}$$

This form of the solution obscures the simplicity of the dynamics in this case. We will discuss this in the Swift Method section.

Real Jordan Canonical Form

Following Perko, we can define $P_r = [\mathbf{v}_1\ \mathbf{u}_1\ \mathbf{v}_2\ \mathbf{u}_2] \in \mathbb{R}^{4\times 4}$ and get a standard form of the matrices: In case 1

$$P_r^{-1}AP_r = \begin{bmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix},$$

and in case 2

$$P_r^{-1}AP_r = \begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix}.$$

The Swift Method

We have previously discussed two shortcuts for computing the solution to $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ that I have humbly dubbed the "Swift Method". They apply to $A \in \mathbb{R}^{2\times 2}$ with eigenvalues $a \pm ib, b > 0$, and also to $A \in \mathbb{R}^n$ with a single (real) eigenvalue with multiplicity n.

With complex eigenvalues, the general solution can be written as

$$\mathbf{x}(t) = e^{at}\big(\mathbf{x}_0\cos(bt) + \mathbf{y}_0\sin(bt)\big), \quad \text{where } \mathbf{y}_0 = \tfrac{1}{b}(A - aI_2)\mathbf{x}_0$$

This works because $\mathbf{y}_0 + i\mathbf{x}_0$ is an eigenvector with eigenvalue $a + ib$ for any $\mathbf{x}_0 \in \mathbb{R}^2$. Note that this trick will not work if A is larger than 2×2 and there is another eigenvalue besides $a \pm ib$.

With repeated real eigenvalues, the general solution is

$$\mathbf{x}(t) = e^{\lambda t}\left(\mathbf{x}_0 + t\mathbf{x}_1 + \frac{t^2}{2!}\mathbf{x}_2 + \frac{t^3}{3!}\mathbf{x}_3 + \cdots + \frac{t^{n-1}}{(n-1)!}\mathbf{x}_{n-1}\right),$$

where $\mathbf{x}_j = (A - \lambda I_n)\mathbf{x}_{j-1}$ for $j \in \{1, 2, \ldots n-1\}$. Note that the solution is correct regardless of the Jordan canonical form of A. For example, if $\delta_1 = n$ then any nonzero $\mathbf{x}_0$ is an eigenvector and $\mathbf{x}_1 = \mathbf{0}$. Thus, the solution is $\mathbf{x}(t) = \mathbf{x}_0 e^{\lambda t}$. This method works because $\delta_n = n$, and any vector $\mathbf{x}_0 \in \mathbb{R}^n$ satisfies $(A - \lambda I_n)\mathbf{x}_0 = \mathbf{0}$. In other words, this method works because any $\mathbf{x}_0 \in \mathbb{R}^n$ is a generalized eigenvector with eigenvalue λ. Note that this trick will not work if there is another eigenvalue besides λ.

In this section, we combine these two techniques, to find a solution for the case where $A \in \mathbb{R}^{2n\times 2n}$ has eigenvalues $a \pm ib, b > 0$ with multiplicity n. We let δ_k be the deficiency index for $\lambda = a + ib$. I do not have a formal proof that the algorithm works, but it can be computed with symbolic MATLAB and the solution can be checked.

For simplicity, we first assume that with $A \in \mathbb{R}^{4\times 4}$ has eigenvalues $a \pm ib$ with multiplicity 2. We search for a simple-to-compute solution to $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$.

Suppose we are in Case 2, where A is semisimple. To better understand the dynamics, consider the solution in Equation (4) with $r_1 = 1, \theta_1 = 0$, and $r_2 = 0$:

$$\mathbf{x}(t) = e^{at}\Big(\mathbf{v}_1\cos(bt) + \mathbf{u}_1\sin(bt)\Big). \tag{3}$$

This solution has $\mathbf{x}(0) = \mathbf{v}_1$. Fortunately, we can find an eigenvector for $\lambda = a + ib$ with any given imaginary part. The eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$, with $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$, for $\lambda = a + ib$ satisfies

$$\mathbf{u} = \tfrac{1}{b}(A - aI_4)\mathbf{v}, \quad \mathbf{v} = -\tfrac{1}{b}(A - aI_4)\mathbf{u}.$$

Thus, the condition that $\mathbf{u} + i\mathbf{v}$ is an eigenvector or A is that Note that $\mathbf{u}$ and $\mathbf{v}$ satisfy

$$\mathbf{u} = \frac{-1}{b^2}(A - aI_4)^2\mathbf{u}, \quad \mathbf{v} = \frac{-1}{b^2}(A - aI_4)^2\mathbf{v}.$$

A theorem from linear algebra says that A satisfies $(A - aI)^2 + b^2 I = 0$, since the minimal polynomial of A is $(\lambda - a)^2 + b^2$. Thus, $\mathbf{u}$ and $\mathbf{v}$ satisfy $\mathbf{u} = I_4\mathbf{u}$ and $\mathbf{v} = I_4\mathbf{v}$. In other words, any nonzero vector in $\mathbb{R}^4$ can be the real or the imaginary part of an eigenvector.

Given $\mathbf{x}_0 \neq \mathbf{0}$, one eigenvector of A with the simple eigenvalue $a+ib$ is $\mathbf{u}_1 + i\mathbf{v}_1 = \frac{1}{b}(A - aI_4)\mathbf{x}_0 + i\mathbf{x}_0$, and we do not need to find a second eigenvector to use Equation (3). The solution to $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{at}\left(\mathbf{x}_0\cos(bt) + \tfrac{1}{b}(A - aI_4)\mathbf{x}_0\sin(bt)\right). \tag{4}$$

As a consequence, for any $\mathbf{x}_0 \neq \mathbf{0}$, the ODE $\dot{\mathbf{x}} = A\mathbf{x}$ has an invariant subspace spanned by $\mathbf{x}_0$ and $\frac{1}{b}(A - aI_4)\mathbf{x}_0$. The solution to the ODE with $\mathbf{x}(0) = \mathbf{x}_0$ is an ellipse is this 2-dimensional subspace. Furthermore, the matrix exponential is

$$\exp(At) = e^{at}\left(I_4\cos(bt) + \tfrac{1}{b}(A - aI_4)\sin(bt)\right).$$

Example: Consider the ODE

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x}.$$

In this case, A is the real Jordan Canonical form for a 4×4 matrix with eigenvalues $\lambda = \pm i$ and $\delta_1 = 2$. Since $a = 0$ and $b = 1$, the solution to $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x}(t) = \mathbf{x}_0\cos(t) + A\mathbf{x}_0\sin(t)$. Consider the 3-sphere $\{\mathbf{x} \in \mathbb{R}^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$. Through every point on the 3-sphere there is a circle which is the trajectory of this ODE. The ODE has a unique solution, so no two trajectories intersect. The collection of all such trajectories (which are circles) is called the Hopf fibration of the 3-sphere. $\diamond$

Next, assume that $A \in \mathbb{R}^{4\times 4}$ has eigenvalues $a \pm ib$ with multiplicity 2, and $\delta_1 = 1$. That is, A is not semisimple and we are in Case 1. The solution (1) with $r_1 = 0, r_2 = 1$, and $\theta_2 = 0$ is

$$\mathbf{x}(t) = e^{at}\Big((\mathbf{v}_2 + t\mathbf{v}_1)\cos(bt) + (\mathbf{u}_2 + t\mathbf{u}_1)\sin(bt)\Big). \tag{5}$$

Note that $\mathbf{x}(0) = \mathbf{v}_2$. Thus, we search for a solution to $(A - (a + ib)I_4)^2(\mathbf{u}_2 + i\mathbf{v}_2) = \mathbf{0}$, with $\mathbf{v}_2 = \mathbf{x}_0$. To this end, we define

$$A_r = Re((A - (a + ib)I_4)^2), \quad A_i = Im((A - (a + ib)I_4)^2). \tag{6}$$

The complex equation $(A_r + iA_i)(\mathbf{u}_2 + i\mathbf{v}_2) = \mathbf{0}$ becomes

$$A_r\mathbf{u}_2 - A_i\mathbf{v}_2 = \mathbf{0}, \quad A_i\mathbf{u}_2 + A_r\mathbf{v}_2 = \mathbf{0}.$$

We focus on the second equation, and solve for $\mathbf{u}_2$ in terms of $\mathbf{v}_2$:

$$\mathbf{u}_2 = -A_i^{-1}A_r\mathbf{v}_2. \tag{7}$$

We can get a generalized eigenvector We can set $\mathbf{v}_2 = \mathbf{x}_0$ and get a generalized eigenvector $\mathbf{y}_0 + i\mathbf{x}_0$ of rank 2 using Equation (7). (I have a sketch of a proof, but it is too long to fit in this margin.) Then we can get a true eigenvector of rank 1, denoted $\mathbf{y}_1 + i\mathbf{x}_1$ as defined in the following algorithm.

The Swift method for repeated complex eigenvalues: (the 4×4 case)
Assume $A \in \mathbb{R}^{4\times 4}$ has eigenvalues $a \pm ib$ with multiplicity 2, and $\mathbf{x}_0 \in \mathbb{R}^4$.

- Define $A_\lambda = A - (a+ib)I_4$, $A_r = \text{real}(A_\lambda^2)$, $A_i = \text{imag}(A_\lambda^2)$.
- Compute $\mathbf{y}_0 = -A_i^{-1}A_r\mathbf{x}_0 \in \mathbb{R}^4$.
- Find $\mathbf{x}_1, \mathbf{y}_1 \in \mathbb{R}^4$ using $\mathbf{y}_1 + i\mathbf{x}_1 = A_\lambda(\mathbf{y}_0 + i\mathbf{x}_0)$.
- The solution to $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is
  $\mathbf{x}(t) = e^{at}\Big(\big(\mathbf{x}_0 + \mathbf{x}_1 t\big)\cos(bt) + \big(\mathbf{y}_0 + \mathbf{y}_1 t\big)\sin(bt)\Big)$.

In case 1 ($\delta_1 = 1$), note that $\mathbf{y}_0 + i\mathbf{x}_0$ is automatically a generalized eigenvector, since $A_\lambda^2(\mathbf{y}_0 + i\mathbf{x}_0) = \mathbf{0}$. In general, $\mathbf{y}_0 + i\mathbf{x}_0$ is a rank 2 generalized eigenvector, but it might also satisfy $A_\lambda(\mathbf{y}_0 + i\mathbf{x}_0) = \mathbf{0}$ in which case it is an eigenvector (with rank 1).

In case 2 ($\delta_1 = 2$), note that $\mathbf{y}_0 + i\mathbf{x}_0$ is automatically an eigenvector since A has no rank 2 generalized eigenvectors. We found $\mathbf{y}_0$ by setting $A_\lambda^2(\mathbf{y}_0 + i\mathbf{x}_0) = \mathbf{0}$ but we could have also solved $A_\lambda(\mathbf{y}_0 + i\mathbf{x}_0) = \mathbf{0}$ and gotten the same $\mathbf{y}_0$. If we know that $\delta_1 = 2$, then there is no need to compute the real and imaginary parts of A_λ^2. The following algorithm takes this observation into account, and allows matrices larger than 4×4.

The Swift method for repeated complex eigenvalues: (the general case)
Assume $A \in \mathbb{R}^{2n\times 2n}$ has eigenvalues $a \pm ib$ with multiplicity n, and $\mathbf{x}_0 \in \mathbb{R}^{2n}$.

- Define $A_\lambda = A - (a+ib)I_{2n}$.
- let $m = n$, or if desired let m be the smallest k such that $\delta_k = n$.
- Let $A_r = \text{real}(A_\lambda^m)$, $A_i = \text{imag}(A_\lambda^m)$.
- Compute $\mathbf{y}_0 = -A_i^{-1}A_r\mathbf{x}_0 \in \mathbb{R}^{2n}$.
- For $j \in \{1, \ldots, m-1\}$, find $\mathbf{x}_j, \mathbf{y}_j \in \mathbb{R}^{2n}$ using $\mathbf{y}_j + i\mathbf{x}_j = A_\lambda(\mathbf{y}_{j-1} + i\mathbf{x}_{j-1})$.
- The solution to $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{at}\left(\left(\mathbf{x}_0 + \mathbf{x}_1 t + \cdots + \mathbf{x}_{m-1}\frac{t^{m-1}}{(m-1)!}\right)\cos(bt) + \left(\mathbf{y}_0 + \mathbf{y}_1 t + \cdots + \mathbf{y}_{m-1}\frac{t^{m-1}}{(m-1)!}\right)\sin(bt)\right).$$

The second step of algorithm gives the option of computing the largest possible rank of a generalized eigenvector. If the algorithm is automated, then setting $m = n$ is simpler. However, setting m to the smallest k such $\delta_k = n$ gives more insight into the geometry of the solution.

Example: If $A \in \mathbb{R}^{2n\times 2n}$ is semisimple with eigenvalues $a \pm ib$, then $\delta_1 = n$ and $m = 1$. Then $A_r = A - aI_{2n}$ and $A_i = -bI_{2n}$. In this case, $\mathbf{y}_0 = \frac{1}{b}(A - aI_{2n})\mathbf{x}_0$, and $\mathbf{y}_0 + i\mathbf{x}_0$ is an eigenvector. Note that we do not need to find n linearly independent eigenvectors of A to solve $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$. The solution is simply

$$\mathbf{x}(t) = e^{at}\big(\mathbf{x}_0\cos(bt) + \mathbf{y}_0\sin(bt)\big), \quad \text{where } \mathbf{y}_0 = \tfrac{1}{b}(A - aI_{2n})\mathbf{x}_0.$$

This reduces to the original Swift method for 2×2 matrices ($n = 1$) with eigenvalues $\lambda = a \pm ib$.