The Period Doubling Operator
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The doubling operator $\mathcal{T}$ maps a function f to a new function $\mathcal{T}f$, which can also be denoted $\mathcal{T}(f)$. In my slightly unorthodox definition, the domain $\mathcal{D}$ of $\mathcal{T}$ is the infinite-dimensional space of continuous functions $f : [-1, 1] \to [-1, 1]$ such that

- $f(-1) = f(1) = -1$.
- f has a unique critical point c, and $f''(c) < 0$.
- f has exactly 2 fixed points, -1 and $x^* > c$.

Note that $f \in \mathcal{D}$ is increasing on $[-1, c]$ and decreasing on $[c, 1]$, and $f(c) > c$. There is a unique $x_\ell < c$ such that $f(x_\ell) = x^*$. Define the function $h : [x_\ell, x^*] \to [-1, 1]$ as the unique linear function that satisfies $h(x^*) = -1$ and $f(x_\ell) = 1$. One formula is $h(x) = -1 + 2\frac{x-x^*}{x_\ell - x^*}$.

The doubled function $\mathcal{T}f : [-1, 1] \to \mathbb{R}$ is defined by $\mathcal{T}f = h \circ f^2 \circ h^{-1}$.

An important special case is when f is even. In this case $c = 0$ and $x_\ell = -x^*$, so $h(x) = -\frac{x}{x^*}$ and $h^{-1}(x) = -xx^*$. Therefore, $\mathcal{T}f(x) = -\frac{1}{x^*}f^2(xx^*)$. It is traditional in this case to define $\alpha = 1/x^*$, so

$$\text{if } f \in D \text{ is even, and } f(1/\alpha) = 1/\alpha > 0, \text{ then } \mathcal{T}f(x) = -\alpha f^2(x/\alpha).$$

The Feigenbaum conjectures, proved by Oscar Lanford, are

- $\mathcal{T}$ has a unique fixed point in $\mathcal{D}$, which is even, denoted by g.
- The linearization of $\mathcal{T}$ at g, $D\mathcal{T}(g)$, has exactly one eigenvalue outside the unit circle. This eigenvalue is $\delta \approx 4.669$. Furthermore, there are no eigenvalues on the unit circle.

This explains why the *Feigenbaum constant* $\delta \approx 4.669$ is a "universal number". There is another universal number $\alpha \approx 2.503$ defined as the reciprocal of the unique fixed point of g in $[0, 1]$. The "universal function" g satisfies $g(x) = -\alpha g^2(x/\alpha)$.

It should be mentiond that another paper independently discovered this universality: Coullet, P; Tresser, C (1978), "Iteration d'endomorphismes et groupe de renormalisation", J. Phys. Colloque, 539: 525. Predrag Cvitanović also deserves mention.

Application to the logistic map family $f_a(x) = -1 + \frac{a}{2}(1 - x^2)$, with $0 \le a \le 4$:
The logistic map f_a is in the domain of $\mathcal{T}$ iff $2 < a \le 4$.
The doubled map $\mathcal{T}f_a$ is itself in the domain of $\mathcal{T}$ iff $a_2 < a \le M$, where $a_2 = 1 + \sqrt{5}$, so that f_{a_2} has a superstable period 2 orbit, and $M \approx 3.67857$ is the first Misiurewicz point, where f_M has the 2 to 1 band merging.
The only map in the family f_a such that $\mathcal{T}^n f_a \in \mathcal{D}$ for all n is the one with $a = a_\infty$, at the accumulation of period doublings. Furthermore,

$$\lim_{n\to\infty} \mathcal{T}^n f_{a_\infty} = g,$$

the universal function.