MAT 667 (Dynamical Systems)
Homework # 1 due Wednesday, January 25, 2017, in class.

1. When solving ODEs with pencil and paper, it is often preferable to leave parameters in the problem. For example, the solution to $\frac{dx}{dt} = kx$, $x(0) = x_0$ is $x(t) = x_0 e^{kt}$. However, when solving ODEs numerically, we need to explicitly choose the parameters. Thus it is best to scale the problem as much as possible before approaching the computer.

Show that, but a suitable scaling of time $t = \alpha \bar{t}$ and positon $x = \beta \bar{x}$, we can write any initial value problem $\frac{dx}{dt} = kx$, $x(0) = x_0$, with k and x_0 nonzero, in the form

$$\frac{d\bar{x}}{d\bar{t}} = \bar{x}, \quad \bar{x}(0) = 1.$$

Solve this last IVP by inspection and then use the scaling to recover the known general solution. If we needed to solve the ODE using numerical methods, we would only need to solve one ODE instead of lots of cases with different choices of k and x_0.

Show that, if we restrict α to be positive, so that we do not reverse the direction of time, then there are two ODEs to solve: $\frac{d\bar{x}}{d\bar{t}} = \bar{x}$ and $\frac{d\bar{x}}{d\bar{t}} = -\bar{x}$.

2. The driven, damped pendulum has this equation of motion $F = ma$, actually $ma = F$, for the tangential acceleration:

$$m\ell \frac{d^2\theta}{dt^2} = -\gamma \frac{d\theta}{dt} - mg\sin(\theta) + F\cos(\omega_d t).$$

The unit of mass m is kilograms. We write this statement as $[m] = \text{kg}$. Similarly, $[\ell] = \text{m}$ (meters), $[t] = \text{s}$ (seconds). This is the MKS system: (meters, kilograms, seconds). Angles like θ, measured in radians, are unitless, so $[\theta] = 1$. The acceleration of gravity g has units $[g] = \text{m/s}^2$. Define $\omega_0 = \sqrt{g/\ell}$. Find the units of ω_0, and write the equation of motion as

$$\frac{d^2\theta}{dt^2} = -c\omega_0 \frac{d\theta}{dt} - \omega_0^2 \sin(\theta) + \rho\omega_0^2 \cos(\omega_d t).$$

Find the constants c and ρ in terms of the original constants m, ℓ, γ, g, and F, and find the units $[c]$ and $[\rho]$. This is the equation of motion solved by the *Mathematica* program. Now do a change of the time variable by defining $\omega_0 t = \bar{t}$. Show that the equation of motion becomes

$$\frac{d^2\theta}{d\bar{t}^2} = -c\frac{d\theta}{d\bar{t}} - \sin(\theta) + \rho\cos(\bar{\omega}\bar{t}).$$

Find the new constant $\bar{\omega}$ in terms of ω_d and ω_0, the driving frequency and natural frequency.

3. The chaos demonstration made by Doug LaMaster, Jeremy Petak, and Sam Zerbib has a marble rolling on a washboard of the shape $y = -H\cos(\pi x/L)$ shaken back and forth in the x-direction with position $A\cos(\omega t)$. If $x(t)$ is the position of the marble relative to the washboard, Newton's laws of motion are approximately

$$\frac{7}{5}m\frac{d^2x}{dt^2} = -\frac{mgH\pi}{L}\sin\left(\frac{\pi x}{L}\right) - m\gamma\frac{dx}{dt} + mA\omega^2\cos(\omega t).$$

(The factor of $\frac{7}{5}$ is due to the rolling (as opposed to sliding) of the marble. The first forcing term is from gravity acting on the slope $\frac{dy}{dx}$ in the small angle approximation assuming $H/L \ll 1$. The frictional term has a force proportional to the velocity; the friction constant γ has units of inverse time. The last term is a "pseudo-force" obtained because the equations are written in the reference frame of the accelerating washboard.)

It is clear that the mass m of the marble cancels in the equation of motion. Show that the time scaling $t = \alpha\bar{t}$, with an appropriately chosen α, gives the driven, damped pendulum equation for the variable $\theta = \pi x/L$:

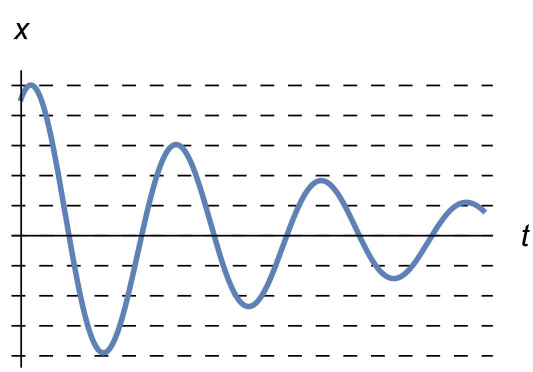
$$\frac{d^2\theta}{d\bar{t}^2} = -\sin(\theta) - c\frac{d\theta}{d\bar{t}} + \rho\cos(\bar{\omega}\bar{t}),$$

Find the dimensionless parameters c, ρ, and $\bar{\omega}$ in terms of the dimensional parameters H, L, A, γ, ω, and g.

4. The constant c is the hard one to determine in the driven damped pendulum equation as applied to the shaken washboard. We can estimate c by looking at the approach of the marble to the bottom of the well when the washboard is stationary ($A = 0$). The scaled ODE in this case, with $|\theta|$ small, is approximated by

$$\frac{d^2\theta}{d\bar{t}^2} + c\frac{d\theta}{d\bar{t}} + \theta = 0.$$

Find the general solution to this ODE and use the solution to estimate c if the position of the marble approaches 0 in the manner shown when x is small compared to L.



Hint: After one quasi-period the amplitude is decreased by a factor of about 3/5.