MAT 667 (Dynamical Systems), Prof. Jim Swift
Take-Home Final Exam, Spring 2023
This version version 2, updated May 5.

The exam is due Monday, May 8, at noon. Send me a pdf via email, or bring a paper solution to my office. (Slip the paper under my door if I'm not there.)

Groundrules: Treat this like a homework assignment. You may work with other students in the class, but turn in your own work. You may use any books, notes, computer programs, calculators, or information on the internet.

1. Consider the map $f_a : [0, \infty) \to [0, \infty)$, defined by $f_a(x) = \dfrac{ax}{1 + x^2}$. The parameter a is positive.

(a) The trivial fixed point is $x = 0$ for all a. Determine the stability of the trivial fixed point as a function of a.

(b) Find the nontrivial fixed point, as a function of a, and determine the values of a for which a nontrivial fixed point exists.

(c) Determine the stability of the nontrivial fixed point as a function of a. (That is, find an expression for $f_a'(x)$, evaluated at the nontrivial fixed point, and write this expression purely in terms of a.)

(d) Find the value of a for which the nontrivial fixed point is superstable.

(e) Sketch the bifurcation diagram in the a-x plane, indicating stable solution branches with solid lines and unstable solution branches with dotted lines.

2. Consider the iterated map $x_{n+1} = f_a(x_n)$, where $f_a : [-1, 1] \to [-1, 1]$ is defined by $f_a(x) = a(x - x^3)$, with a real parameter a (positive or negative) with $|a| \leq 3\sqrt{3}/2$. Note that $f_a(x)$ has critical points at $x = \pm 1/\sqrt{3}$, and$f_a(\pm 1/\sqrt{3}) = \pm 2a/(3\sqrt{3})$.

(a) Find the fixed points as a function of a. (Note that the fixed points have $x^2 \leq 1$.)

(b) Find the stability of the fixed points you found in part (a). Do not resolve ambiguity when linearization is inconclusive.

(c) Find the period 2 points which satisfy $f_a(x) = -x$, $x \neq 0$. Note: you do NOT have to find all the period 2 points.

(d) Find the stability of the period 2 points you found in part (c). Again, do not resolve ambiguity when linearization is inconclusive.

(e) Sketch a bifurcation diagram by hand, showing the fixed points and period 2 points you have found on the a-x plane. Indicate stable points with a solid line, and unstable points with a dotted line.

3. Consider the Partial Differential Equation (PDE) for the function $(x,t) \mapsto u(x,t)$,

$$\frac{\partial u}{\partial t} = \alpha u - \beta u^3 + \gamma \frac{\partial^2 u}{\partial x^2}$$

where α, β, and γ are positive constants. This is called a reaction-diffusion equation when x is position and t is time. Consider the scaling $t = a\bar{t}$, $x = b\bar{x}$ and $u = c\bar{u}$, with the positive constants a, b, and c. Find a, b, and c as functions of α, β, and γ so that

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \bar{u} - \bar{u}^3 + \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}.$$

4. Consider the standard map, $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$, where

$$f(x,y) = \left(x + y - \frac{k}{2\pi}\sin(2\pi x), y - \frac{k}{2\pi}\sin(2\pi x) \right)$$

and k is a positive constant. Note that $(\frac{1}{2}, 0)$ is a fixed point for every value of k. Find the eigenvalues of $Df(\frac{1}{2}, 0)$ as a function of $k > 0$ and determine stability of this fixed point.

5. Let $\omega > 0$ be a constant. Find the periodic particular solution to

$$\frac{d^3 x}{dt^3} = x + \cos(\omega t).$$

Write the particular solution in two forms, $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$ and $x_p(t) = R\cos(\omega t - \delta)$.

That is, find A, B, R, and δ in terms of ω.

6. Consider the iterated map of the plane, $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$, where

$$f(x,y) = (x(1-y), x - y + 1)$$

(a) Find the fixed point(s) of the map.

(b) Linearize the map about the fixed point(s). Find the eigenvalues of the linearized system(s). Using the eigenvalues, determine if the fixed point(s) are stable or unstable, or indicate that the linearization does not give enough information to tell.