Response of a Driven Damped Linear Oscillator

Dr. James W. Swift, NAU

We seek a particular solution $x_p : \mathbb{R} \to \mathbb{R}; t \mapsto x_p(t)$ to the ODE

$$\frac{d^2x}{dt^2} + \frac{1}{Q}\frac{dx}{dt} + x = \cos(\omega t), \tag{1}$$

where $Q > 0$ and $\omega > 0$ are real parameters. We look for a solution of the form

$$x_p(t) = \mathrm{Re}(\hat{A}e^{i\omega t}) = R\cos(\omega t - \delta) = A\cos(\omega t) + B\sin(\omega t), \tag{2}$$

where $\hat{A} \in \mathbb{C}$ and $\hat{A} = Re^{-i\delta} = A - iB$, for $R, \delta, A, B \in \mathbb{R}$. The ODE (1) is the real part of

$$\frac{d^2}{dt^2}\left[\hat{A}e^{i\omega t}\right] + \frac{1}{Q}\frac{d}{dt}\left[\hat{A}e^{i\omega t}\right] + \hat{A}e^{i\omega t} = e^{i\omega t}.$$

Using the facts that $\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t}$ and $\frac{d}{dt}\mathrm{Re}\left(e^{i\omega t}\right) = \mathrm{Re}\left(\frac{d}{dt}e^{i\omega t}\right)$, the previous equation becomes

$$\hat{A}\left[-\omega^2 + i\omega/Q + 1\right]e^{i\omega t} = e^{i\omega t}.$$

This equation needs to be true for all t, so

$$\hat{A} = \frac{1}{(1-\omega^2) + i\omega/Q} = \frac{(1-\omega^2) - i\omega/Q}{(1-\omega^2)^2 + (\omega/Q)^2}.$$

The amplitude of $\hat{A}$ is

$$R = |\hat{A}| = \frac{1}{\sqrt{(1-\omega^2)^2 + (\omega/Q)^2}} = \frac{Q}{\sqrt{Q^2(1-\omega^2)^2 + \omega^2}},$$

so $\hat{A} = R^2\left((1-\omega^2) - i\omega/Q\right)$. Note that $\hat{A} = Re^{-i\delta} = A - iB$ is in the third or fourth quadrant since $B > 0$. Consequently, $0 < \delta < \pi$. Since $\tan(\pi/2)$ is undefined it not advisable to define δ in terms of arctangent. We can write δ, A, and B in terms of R:

$$\delta = \arccos(R(1-\omega^2)), \quad A = R^2(1-\omega^2), \quad B = R^2\omega/Q$$

These can now be used in Equation (2) to find the periodic particular solution to (1). Note that when $\omega = 1$, we get $R = Q$, $\delta = \pi/2$, $A = 0$, and $B = Q$. Thus, the particular solution is $x_p(t) = Q\sin(t)$ when $\omega = 1$.

Here are some results about the resonance curve when Q is large. The maximum value of R is approximately Q, and occurs at $\omega \approx 1$. The full width at half maximum power is approximately $1/Q$. These results are the reason why high Q oscillators are important in many applications.