MAT 667 (Dynamical Systems)
Homework # 1 (version 3), Spring 2023

1. Let $P(t)$ be the population of some community at time t. The Logistic Model of population growth says that

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{1}$$

for some constants $r > 0$ and $K > 0$. Note that when $P/K \ll 1$, the ODE is approximately $\frac{dP}{dt} \approx rP$, and the population grows exponentially like $P(t) \approx P_0 e^{rt}$ for as long as $P(t)/K \ll 1$. For example, if t is measured in years and a small population grows at 10% per year (compounded continuously) then $r = 0.1$. One can actually write down an explicit formula for the solution to this nonlinear ODE for any initial condition $P(0) = P_0$, but we will not explore this solution since most nonlinear ODEs cannot be solved in closed form. Instead, we will investigate the system numerically and consider the value of scaling the problem to decrease the number of parameters.

(a) Suppose you are modeling the population on a small island with $r = 0.1$ and $K = 1000$ and t measured in years. Use Darryl Nester's slope field app to investigate this system for these parameters $r = 0.1$ and $K = 1000$. Choose the variables $\frac{dx}{dt}$ in the app. Use the "gear" tool to plot $0 \leq t \leq 100$ and $0 \leq x \leq 1200$. Note the two constant solutions, $P(t) = 0$ and $P(t) = 1000$.

(b) Investigate the effect of changing the parameters r and K. For example, try $r = 0.05$ and $r = 0.2$. Also change to $K = 900$ and $K = 1100$.

(c) (Turn this in.) Do a scaling Equation (1), by scaling the variables $P = \alpha\bar{P}$ and $t = \beta\bar{t}$. Chose the constants α and β to put the ODE in the form

$$\frac{d\bar{P}}{d\bar{t}} = \bar{P}(1 - \bar{P}). \tag{2}$$

Solve for the dimensionless time $\bar{t}$, and the scaled population $\bar{P}$ in terms of the original variables and the original parameters. Write a few sentences to describe the significance of $\bar{t}$ and $\bar{P}$.

(d) Note that the scaled ODE (2) has no free parameters, so we only need to solve a single ODE. (We do need to consider different initial conditions.) Go back to the slope field app, and investigate that scaled ODE.

(e) (Turn this in.) Make a sketch of several solutions to the scaled ODE, or include the computer-generated figure like I did for Problem 3. Write a caption for the figure, as if it were a journal article.

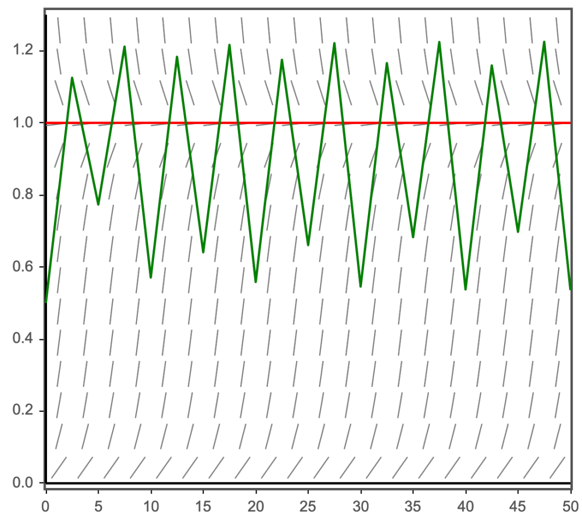
All of the rest of the problems describe something to turn in.

2. This Logistic Model is related to the Logistic Map that we have been studying. One way to approximate the solutions to Equation (2) is Euler's method. Fix a step size $h > 0$. Euler's method gives an approximation $\bar{P}_n$ for the solution to the ODE at time $\bar{t} = nh$. Given the initial condition $\bar{P}_0$, the approximate solution is the sequence of the iterated map

$$\bar{P}_{n+1} = \bar{P}_n + h \cdot \bar{P}_n(1 - \bar{P}_n).$$

Do a scaling $\bar{P}_n = \alpha x_n$, and choose α to make x_n satisfy the Logistic Map $x_{n+1} = a x_n(1 - x_n)$. Find a in terms of h. Show that the fixed point of the logistic map at any $a \in (1, 4]$ scales to $\bar{P} = 1$.

3. Use the slope field app again, to investigate Equation (2). But this time use Euler's method with outrageously large step sizes, for example $h = 2$ or 3. Write a caption for this figure, which shows two approximate solutions to Equation (2), using Euler's method with step size $h = 2.5$. Describe the relation to the Logistic map.



4. The linear second order ODE for a mass on a spring with friction is

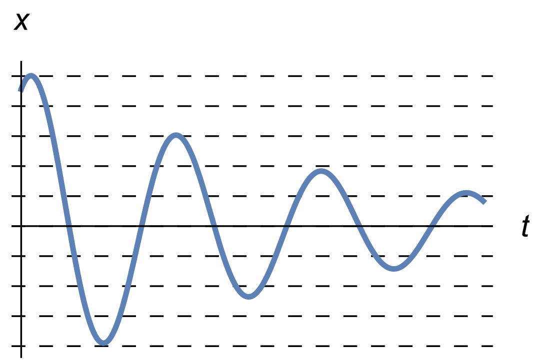
$$m \frac{d^2 x}{dt^2} = -kx - \gamma \frac{dx}{dt} \tag{3}$$

where $m > 0$ is the mass, $k > 0$ is the spring constant, and $\gamma > 0$ is the friction constant. Show that, by an appropriate scaling $t = \alpha \bar{t}$, with $\alpha > 0$, this becomes

$$\frac{d^2 x}{d\bar{t}^2} = -x - c \frac{dx}{d\bar{t}} \tag{4}$$

where c is a dimensionless friction constant. Find c in terms of m, k, and γ.

5. Convert the ODE (4) to a system of 2 first order ODEs and solve with Darryl Nester's Slope Field App. Play with various values of the damping parameter c, and estimate the value of c so that this graph is a solution to the differential equation (3), with the original t:



Hints: Note that no scale on the axes is shown, or needed. The important thing is that after one oscillation the amplitude is decreased by a factor of about 3/5. Thus, you can answer the question by investigating Equation (4). This graph might show oscillations in a galaxy where the time axis spans billions of years. Or the graph might show oscillations in a molecule where the time axis spans nanoseconds.

6. The driven, damped pendulum is a very important dynamical system. It has this equation of motion $F = ma$, actually $ma = F$, for the tangential acceleration:

$$m\ell \frac{d^2\theta}{dt^2} = -mg\sin(\theta) - \gamma\frac{d\theta}{dt} + F\cos(\omega_d t).$$

The unit of mass m is kilograms. We write this statement as $[m] = \text{kg}$. Similarly, $[\ell] = \text{m}$ (meters), $[t] = \text{s}$ (seconds). This is the MKS system: (meters, kilograms, seconds). Angles are special; $[\theta] = 1$ since radians are dimensionless. Figure out the units of γ, g, F, and ω_d so that the units of each term in the ODE are the same.

Do a scaling of time, $t = \alpha\bar{t}$ and choose α so the equations of motion become

$$\frac{d^2\theta}{d\bar{t}^2} = -\sin(\theta) - c\frac{d\theta}{d\bar{t}} + \rho\cos(\omega\bar{t}).$$

Find the three essential constants c, ρ and ω in terms of the six original constants m, ℓ, γ, g, and F, and ω_d, and show that all three constants (as well as $\bar{t}$ and θ) are dimensionless.

Even with "only" 3 parameters instead of 6, understanding the behavior of solutions to the driven damped pendulum for all choices of parameters (and initial conditions) is a seemingly impossible task.