MAT 667 (Dynamical Systems)
Homework # 2, Spring 2023 version 4

1. Newton's Law of cooling for an object in an oscillating ambient temperature is

$$\frac{dx}{dt} = -k(x - (A + B\cos(\omega t)).$$

See our website for February 17, 20. We found that the scaled form of the ODE is

$$\frac{d\bar{x}}{d\bar{t}} = -K(\bar{x} - \cos(\bar{t})),$$

where $\bar{x} = (x - A)/B$, $\bar{t} = \omega t$, and $K := k/\omega$. Drop the hats and consider the scaled ODE

$$\frac{dx}{dt} = -K(x - \cos(t))$$

The general solution to this ODE is $x(t) = x_h(t) + x_p(t)$, as discussed in class.

Find the general solution to the scaled equation, using the complex variable technique of $x_p(t) = \text{Re}(e^{it})$. Write the solution in both the form $x_p(t) = A\cos(t) + B\sin(t)$ and $x_p(t) = R\cos(t - \delta)$, getting expressions for A, B, R, and δ in terms of K.

2. Use your results from problem 1 to solve the unscaled IVP

$$\frac{dx}{dt} = -2(x - (3 + 4\cos(5t)), \quad x(0) = 6.$$

3. Recall that the periodic particular solution to $\frac{dx}{dt} + \frac{1}{Q}\frac{dx}{dt} + x = \cos(\omega t)$ is $x_p = R\cos(\omega t - \delta)$, where

$$R = |\hat{A}| = \frac{1}{\sqrt{(1-\omega^2)^2 + (\omega/Q)^2}} = \frac{Q}{\sqrt{Q^2(1-\omega^2)^2 + \omega^2}}.$$

Consider $Q > 0$ as fixed and find the maximum of R as a function of $\omega \geq 0$. Show that the maximum point of the graph $R = f(\omega)$ is $(0, 1)$ for $Q \leq 1/\sqrt{2}$. For $Q > 1/\sqrt{2}$ find the maximum point (ω_*, R_*), which depends on Q.

4. The periodic particular solution to $\frac{d^2x}{dt^2} + \frac{1}{Q}\frac{dx}{dt} + x = \omega^2\cos(\omega t)$ is $x_p = \tilde{R}\cos(\omega t - \delta)$, where

$$\tilde{R} = \frac{\omega^2}{\sqrt{(1-\omega^2)^2 + (\omega/Q)^2}} = \frac{Q\omega^2}{\sqrt{Q^2(1-\omega^2)^2 + \omega^2}}$$

and δ is the same as it is for the ODE in problem 3. (This follows since the ODE is linear, and x_p in problem 4 is ω^2 times x_p in problem 3.)

Let $R = f(\omega)$ as in problem 3. Show that $\tilde{R} = f(1/\omega)$. Use this to find the maximum point $(\omega_*, \tilde{R}_*)$ of the graph $\tilde{R} = \tilde{f}(\omega)$, assuming $Q > 1/\sqrt{2}$, without taking derivatives.

5. The chaos demonstration made by Doug LaMaster, Jeremy Petak, and Sam Zerbib has a marble rolling on a washboard of the shape $y = -H\cos(\pi x/L)$ shaken back and forth in the x-direction with position $A\cos(\omega t)$. If $x(t)$ is the position of the marble relative to the washboard, Newton's laws of motion are approximately

$$\frac{7}{5}m\frac{d^2x}{dt^2} = -\frac{mgH\pi}{L}\sin\left(\frac{\pi x}{L}\right) - m\gamma\frac{dx}{dt} + mA\omega^2\cos(\omega t).$$

(The factor of $\frac{7}{5}$ is due to the rolling (as opposed to sliding) of the marble. The first forcing term is from gravity acting on the slope $\frac{dy}{dx}$ in the small angle approximation assuming $H/L \ll 1$. The frictional term has a force proportional to the velocity; the friction constant γ has units of inverse time. The last term is a "pseudo-force" obtained because the equations are written in the reference frame of the accelerating washboard.)

It is clear that the mass m of the marble cancels in the equation of motion. Show that the time scaling $t = \alpha\bar{t}$, with an appropriately chosen α, gives the driven, damped pendulum equation for the variable $\theta = \pi x/L$:

$$\frac{d^2\theta}{d\bar{t}^2} = -\sin(\theta) - c\frac{d\theta}{d\bar{t}} + \bar{A}\,\bar{\omega}^2\cos(\bar{\omega}\bar{t}),$$

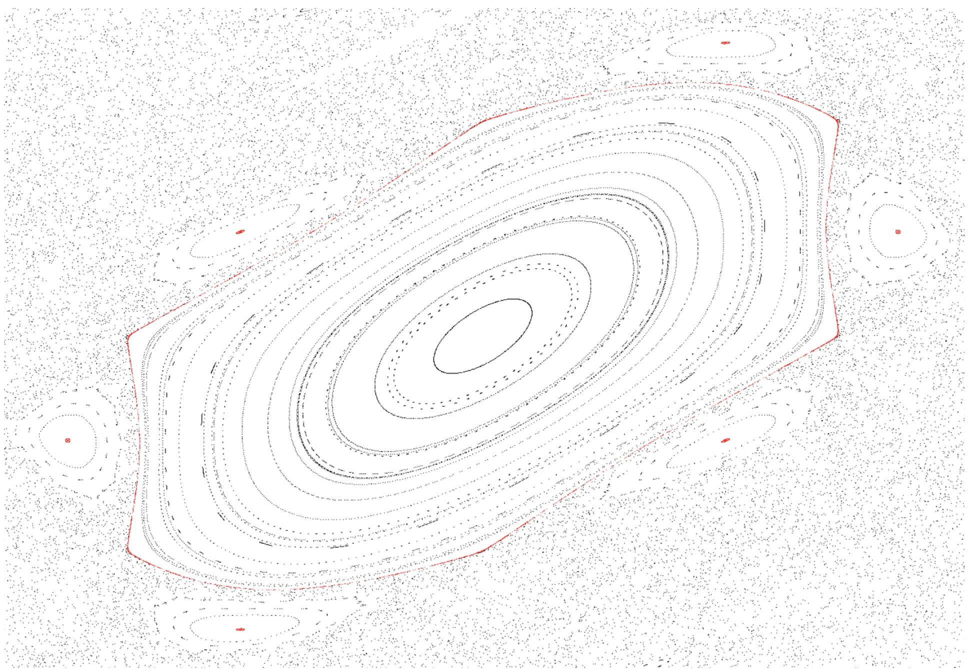
Find the dimensionless parameters c, $\bar{A}$, and $\bar{\omega}$ in terms of the dimensional parameters H, L, A, γ, ω, and g.

Note: This question is very similar to problem 6 in homework 1.

6. The constant c is the hard one to determine in the driven damped pendulum equation as applied to the shaken washboard. We can estimate c by looking at the approach of the marble to the bottom of the well when the washboard is stationary ($A = 0$, or $\omega = 0$). Estimate the dimensionless constant c for the green marble and the yellow ball in the videos posted on the web.

Notes:
(1) This is very open ended. I am looking for your estimate to be within a factor of 2 of my estimate. You may use numerical solutions from the Slope Field and Direction Field Applet, or analytical solutions to the small amplitude ($\sin(x) \approx x$) ODE.
(2) This question is very similar to question 5 in homework 1.

7. The following figure is from the standard map interactive app on our web site. The unstable period 5 solution at the corners of the closed curve is the "k = pi/2, period 6" solution we discussed in class, with the animation on our web site. The stable period 6 solution shown has $k = 1.5708$ ($\pi/2$), and initial condition $x = 4.9697$, $p = 0.7563$, in the notation of the website. The website's x is what I called θ_R (Russian) in class. Translate the website's notation to the θ, ω notation used in the standardMap.nb mathematica notebook. Use the notebook to make an animated gif for this stable period 6 solution.



8. Explore the same interactive app for the standard map used in problem 7. Find another periodic solution that looks interesting and make an animated gif for that periodic solution.