MAT 667 (Dynamical Systems)
Iterated Maps

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1 Introduction

Let X be a metric space with distance function d, $N_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$, and $f : X \to X$. The *orbit* of $x \in X$ is the sequence $(f^n(x))_{n=0}^\infty$ using the notation that $f^0(x) = x$ and $f^n = f \circ f^{n-1}$.

We often use x_0 as the initial point, so the orbit of x_0 is $\{x_0, x_1, x_2, \ldots\}$ which is recursively defined as $x_{n+1} = f(x_n)$ or, eqivalently, $x_n = f^n(x_0)$.

A point p is a *fixed point* of f if $f(p) = p$. A fixed point of f^k which is not a fixed point of f^ℓ for $1 \le \ell < k$ is a *period k point* of f. A *period k orbit* of f is the finite sequence $(p, f(p), \ldots, f^{k-1}(p))$ for some period k point p.

Note that there are k different period k orbits if there is one of them, since the k points need to be distinct. These different period k orbits are all related by cyclic permutations. We sometimes consider the set of them as one periodic orbit.

A fixed point p of f is *stable* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x \in N_\delta(p) \implies f^n(x_0) \in N_\varepsilon(p) \text{ for all } n \in \mathbb{N},$$

A fixed point is *unstable* if it is not stable.

example: Rotation about the origin by angle θ. The origin is a stable fixed point.

A fixed point p of f is *asymptotically stable*, and we say that p is a *sink*, or an *attracting fixed point*, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x \in N_\delta(p) \implies \left(f^n(x) \in N_\varepsilon(p) \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n\to\infty} f^n(x) = p \right).$$

Note that the condition $\lim_{n\to\infty} f^n(x) = p$ alone is true for some fixed points p that do not deserve to be called sinks.

Example: $f : \mathbb{R} \to \mathbb{R}$ defined

$f(x) = x/2$ for $x < 0$, $f(x) = 2x$ for $0 \le x < 1$, and $f(x) = -x$ for $x \ge 1$.

Example: Consider the function $f : S^1 \to S^1$ (the circle), with the lift $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x + \sin^2(2\pi x)$. The circle is $S^1 = \mathbb{R}/\mathbb{Z}$, so $x \sim y$ if $x - y \in \mathbb{Z}$. To be concrete, let S^1 be the interval $[0, 1)$ with the distance function $d(x, y) = \min(|x - y|, 1 - |x - y|)$ and then for any continuous function $F : \mathbb{R} \to \mathbb{R}$ which satisfies $F(x + 1) = F(x) + n$ for all x and for some fixed $n \in \mathbb{Z}$, we get a well-definined continuous circle map $f : S^1 \to S^1$ defined by $f(x) = F(x) - \lfloor F(x) \rfloor$. Note that $F(x + 1) = F(x) + 1$ so th

These examples are shown in the "notSink.nb" notebook on our web site. For a *continuous* map on $\mathbb{R}$, there might not be any such a counter-example, which is perhaps why the "Chaos" textbook gives the simpler definition 1.4

1.1 Monday, April 10

Circle maps are an interesting iterated map, since they share some of the simplicity of maps of $\mathbb{R}$ but the circle has nontrivial topology. For concreteness, define the circle to be $S^1 = [0, 1)$ with the distance function $d(x, y) = \min(|x - y|, 1 - |x - y|)$. Equivalently, $S^1 = \mathbb{R}/\mathbb{Z}$ with the equivalence relation $x \sim y$ if $x - y \in \mathbb{Z}$. The easiest way to define a circle map is to start with its *lift*, a continuous function $F : \mathbb{R} \to \mathbb{R}$ that satisfies $F(x + 1) = F(x) + n$ for all x and some fixed n. Then $f : S^1 \to S^1$ is defined by $f(x) = F(x) - \lfloor F(x) \rfloor$, which we sometimes write as $f(x) = F(x) \pmod 1$.

We call n the *winding number* of the circle map f and its lift F. If F is strictly increasing, and the winding number is 1, then f is orientation preserving and invertible, and f^{-1} is also orientation preserving and invertible. A simple example is $F(x) = x + a$, with $F^{-1}(x) = x - a$. Another example of a continuous, increasing lift with winding number 1 is

$$F(x) = x + a - \frac{b}{2\pi} \sin(2\pi x),$$

with $|b| \le 1$. Note that we cannot get a closed form expression for $F^{-1}(x)$ if $b \neq 0$.

1.2 Wednesday, April 12

A fixed point p of f is *repelling*, and called a *source* if there is an $\varepsilon > 0$ such that for all $x \in N_\varepsilon(p) \setminus \{p\}$ there is an $n \in \mathbb{N}$ such that $f^n(x) \notin N_\varepsilon(p)$.

In other words, all initial conditions x_0 close to the source p have an orbit such that x_n is a distance ε from p for some value of n that depends on x_0.

If $f(p) = p$ and f is C^1, then the fixed point p is a sink if $|f'(p)| < 1$, a source if $|f'(p)| > 1$, and anything can happen if $f'(p) = \pm 1$.

An example of a C^1 function with $f(0) = 0$ and $f'(p) = 1$ is defined by $f(x) = x + x^3 \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$.

1.3 Friday, April 14

Computing the stability of fixed points of maps on $\mathbb{R}^n$.

Definitions of stable, asymptotically stable (attracting) and repelling periodic orbits.

Computing the stability of periodic orbits.

Compute the fixed points and periodic points, and their stability, for the logistic map family