MAT 667 (Dynamical Systems)
Homework # 1.1, Spring 2025 Due Monday, January 27 at 1:40 p.m.

1. A murder is committed, and the police are called. The crime investigator finds the dead body at 3:00 am and its temperature is 30°C, and at 4:00 am the temperature is 29°C. When was the murder committed?

Hints: Assume that the temperature of the body obeys Newton's Law of Cooling, starting at the time of death. Assume that the ambient temperature of the room is 20°C, and that the corpse was 37°C at the time of death. Get an exact answer for the time before 3:00am that the murder occurred, and then round that to the nearest minute and report the estimated time of death.

2. Let $P(t)$ be the population of some community at time t. The Logistic Model of population growth says that

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{1}$$

for some constants $r > 0$ and $K > 0$. Note that when $P/K \ll 1$, the ODE is approximately $\frac{dP}{dt} \approx rP$, and the population grows exponentially like $P(t) \approx P_0 e^{rt}$ for as long as $P(t)/K \ll 1$. For example, if t is measured in years and a small population grows at 10% per year (compounded continuously) then $r = 0.1$. One can actually write down an explicit formula for the solution to this nonlinear ODE for any initial condition $P(0) = P_0$, but we will not explore this solution since most nonlinear ODEs cannot be solved in closed form. Instead, we will investigate the system numerically and consider the value of scaling the problem to decrease the number of parameters.

(a) Suppose you are modeling the population on a small island with $r = 0.1$ and $K = 1000$ and t measured in years. Use Darryl Nester's slope field app to investigate this system for these parameters $r = 0.1$ and $K = 1000$. Choose the variables $\frac{dx}{dt}$ in the app. Use the "gear" tool to plot $0 \leq t \leq 100$ and $0 \leq x \leq 1200$. Note the two constant solutions, $P(t) = 0$ and $P(t) = 1000$.

(b) Investigate the effect of changing the parameters r and K. For example, try $r = 0.05$ and $r = 0.2$. Also change to $K = 900$ and $K = 1100$.

(c) (Turn this in.) Do a scaling Equation (1), by scaling the variables $P = \alpha\bar{P}$ and $t = \beta\bar{t}$. Chose the constants α and β to put the ODE in the form

$$\frac{d\bar{P}}{d\bar{t}} = \bar{P}(1 - \bar{P}). \tag{2}$$

Solve for the dimensionless time $\bar{t}$, and the scaled population $\bar{P}$ in terms of the original variables and the original parameters. Write a few sentences to describe the significance of $\bar{t}$ and $\bar{P}$.

(d) Note that the scaled ODE (2) has no parameters, so we only need to solve a single ODE. (We do need to consider different initial conditions.) Go back to the slope field app, and investigate that scaled ODE.

(e) (Turn this in.) Make a sketch of several solutions to the scaled ODE on the same axes, or download the figure and include it in the pdf or paper submission.

3. The linear second order ODE for a mass on a spring with friction is

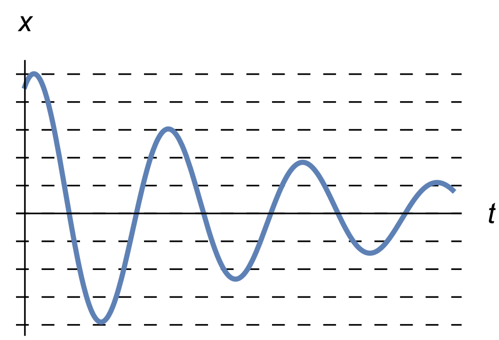
$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} \tag{3}$$

where $m > 0$ is the mass, $k > 0$ is the spring constant, and $\gamma > 0$ is the friction constant. Show that, by an appropriate scaling $t = \alpha\bar{t}$, with $\alpha > 0$, this becomes

$$\frac{d^2x}{d\bar{t}^2} = -x - c\frac{dx}{d\bar{t}} \tag{4}$$

where c is a dimensionless friction constant. Find c in terms of m, k, and γ.

4. Convert the ODE (4) to a system of 2 first order ODEs and solve numerically with Darryl Nester's Slope Field App. Play with various values of the damping parameter c, and estimate the value of c so that this graph is a solution to the differential equation (3):



Hints: (1) Note that no scale on the axes is shown, or needed. The important thing is that after one oscillation the amplitude is decreased by a factor of about 3/5. Thus, you can answer the question by investigating Equation (4). This graph might show oscillations in a galaxy where the time axis spans billions of years. Or the graph might show oscillations in a molecule where the time axis spans nanoseconds.

(2) While the ODE (4) can be solved with pencil and paper, do not use the analytic solution but use the numerical solutions.

5. The Duffing equation is nonlinear, and models a spring which does not satisfy Hooke's Law that the force is proportional to the displacement:

$$m\frac{d^2x}{dt^2} = -kx + sx^3 - \gamma\frac{dx}{dt}$$

where $s > 0$ is a constant that shows how "soft" the spring is. Notice that we did not scale x to get Equation (4). Scale $x = \beta\bar{x}$, for an appropriate β, and use the same time scaling as before, to get the scaled Duffing equation.

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\bar{x} + \bar{x}^3 - c\frac{d\bar{x}}{d\bar{t}} \tag{5}$$