MAT 667 (Dynamical Systems)
Homework # 2, Spring 2025
Due Monday, March 24 at 1:40 p.m.

Note: Use the parameters r, σ, and b for the Lorenz equations. I have changed to this notation at my website, and this is the notation used by all the papers referred to in this homework set. The standard parameters are $r = 28$, $\sigma = 10$, and $b = 8/3$, but for this homework do not assume these standard values.

1. Scan through the 1963 paper by Lorenz, available at our website. Carefully read the sections in Clark Robinson's book "An Introduction to Dynamical Systems: Continuous and Discrete" which are uploaded to Canvas → Files → LorenzByRobinson.pdf.
Verify the calculations on page 246 and 247.

2. Let $\mathbf{r} = (x, y, z)$ and let the Lorenz Equations be written as $\dot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$, for $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$. Compute the matrix $D\mathbf{F}(x, y, x)$, and the divergence of the vector field $(\nabla \cdot \mathbf{F})(x, y, z)$.
Hint: The matrix at the top of page 246 is $D\mathbf{F}(0, 0, 0)$

3. For this problem, pretend you are a physicist, not a mathematician. This problem is another way of thinking about equations (29 - 31) in the 1963 paper by Lorenz.
Let $\mathcal{B}(0)$ be an open, bounded, simply connected region in $\mathbb{R}^3$ with a smooth boundary, $\partial\mathcal{B}(0)$. (Think of a ball: don't worry about topology.) Let $\mathcal{B}(t)$ be the image of $\mathcal{B}(0)$ under the flow of the Lorenz Equations. Let $V(\mathcal{B}(t))$ be the volume of the region $\mathcal{B}(t)$. Convince yourself that the divergence theorem says that

$$\frac{d}{dt}V(\mathcal{B}(t)) = \iint_{\partial\mathcal{B}(t)} \mathbf{F} \cdot \mathbf{dS} = \iiint_{\mathcal{B}(t)} (\nabla \cdot \mathbf{F}) dV.$$

Show that $V(\mathcal{B}(t)) = V(\mathcal{B}(0))e^{-(\sigma+b+1)t}$. This implies that the Lorenz Attractor has zero volume.

4. Find Peter Scott's web page by following the link on the homepage for our course. Click on the picture of the Lorenz attractor (with dots instead of line segments) and get to his page entitled "The Lorenz Attractor" which starts out "Here's a plot produced by our *lorenz* program." That page has the original Lorenz equations (25-27) with variables (X, Y, Z) and in Lorenz's paper, and the scaled Lorenz equations with variables (x, y, z) used by Peter Scott's *lorenz* program.

Show the details of the scaling of the equations that Peter Scott used. So I can follow your work, please start with $X = \alpha x$, $Y = \beta y$, and $Z = \gamma z$. Chose the constants α, β and γ to achieve the scaled equations for $\dot{x}$, $\dot{y}$, and $\dot{z}$.

Find the fixed points of the original equations and the scaled equations. Be explicit about any assumptions you make about r.