MAT 667 (Dynamical Systems), Prof. Swift
Period 3 Implies All Periods

This version was compiled March 12, 2021.

Lemma: If the transition digraph of f has a path $A_1 \to A_2 \to \cdots \to A_k \to A_1$ then f^k has a fixed point c in A_1 with $f^i(c) \in A_{i+1}$ for all $i \in \{1, 2, \ldots, k-1\}$.

Crucial observation of the proof: If f is continuous, $f(A) \supseteq B$ and $f(B) \supseteq C$, then $f^2(A) = f(f(A)) \supseteq f(B) \supseteq C$ for compact intervals A, B, and C. Thus, if there is a path from A to Z in the transition digraph of f, then $f(A) \supseteq Z$.

The tricky part of applying the Lemma is that c might not be a period k point because the closed intervals A_i typically have endpoints in common.

Theorem: Assume the continuous function $f : \mathbb{R} \to \mathbb{R}$ has a period 3 point. For each $k \in \mathbb{N}$, f has a period k point.

Proof:
Let p_1 be a period 3 point of f. Define $p_2 = f(p_1)$ and $p_3 = f(p_2)$. Note the these three period 3 points are distinct. Assume without loss of generality that p_1 is the smallest. We assume $p_1 < p_2 < p_3$, and leave it to the reader to show that the second case, with $p_1 < p_3 < p_2$, is similar.
Define the intervals $A = [p_1, p_2]$ and $B = [p_2, p_3]$. Since f is continuous, $f(A)$ is a closed interval that contains p_2 and p_3. Thus $f(A) \supseteq B$. Similarly $f(B)$ is a closed interval that contains p_3 and p_1, so $f(B) \supseteq A$ and $f(B) \supseteq B$. The transition digraph has (at least) three arrows, $A \to B$, $B \to A$ and $B \to B$ (a loop).

Case 1: $k = 1$. Since $f(B) \supseteq B$, f has a fixed point (a period 1 point).

Case 2: $k = 2$. The path $A \to B \to A$ in the transition digraph shows that $f^2(A) \supseteq A$. By the Lemma, f^2 has a fixed point in A (call it q_1). The lemma also says that $q_2 := f(q_1) \in B$. If q_1 is an endpoint of A, then q_1 is a period 3 point which contradicts $f^2(q_1) = q_1$. Thus $q_1 \in (p_1, p_2)$. It follows that $q_1 \neq q_2$, since $q_2 \in [p_2, p_3]$. Thus q_1 is a period 2 point.

Case 3: $k = 3$. By hypothesis, f has a period 3 point.

Case 4: $k \geq 4$. The Lemma applied to the path $A \to B \to B \cdots \to B \to A$, with $k-1$ B's in a row, says that there is a fixed point (call it q_1) of f^k in A. Furthermore, $q_i := f^{i-1} q_1$ satisfies $q_i \in B$ for $i \in \{2, 3, \cdots, k\}$. If $q_1 = p_2$, the right endpoint of A, then $q_3 = p_1 \notin B$, which is a contradiction of $q_3 \in B$. Thus $q_1 < p_2$ and hence $q_1 \notin B$. Furthermore, q_1 cannot be a fixed point of f^ℓ for any $\ell < k$ since $f^\ell(q_1) = q_{\ell+1} \in B$. Thus, q_1 is a period k point.