MAT 667 (Dynamical Systems), Prof. Swift
Ideas for projects

Updated April 4, 2025.

The projects are meant to be a fairly small exploration, not a Master's thesis. Every person will give a 15-20 minute talk about their project to the class. You may work alone, or in groups of 2 or 3. Ask for permission to have a larger group.

I suggest you make a beamer presentation for your talk. You can ask me for permission to use powerpoint instead.

Send me email about your interests and we can discuss projects. We can talk about possible projects. You don't have to come to me with fully formed ideas. Come to my office hours, or arrange another time.

Anything we have discussed in class can be a starting point for a project. There are also other topics that you might have heard of that can be covered. Here are some examples:

- The Lorenz equations
- The logistic map
- Circle maps
- Complex Dynamics. There are many possible topics here; for example, Julia sets, the Mandelbrot set, and Newton Fractals
- Cellular automata
- The double pendulum
- The Standard Map
- The driven, damped pendulum

Here are some ideas about specific projects: (The ones starting with A are new this semester.)

A. Find a function that approximates the one-dimensional map of $z_n \mapsto z_{n+1} = f(z_n)$ in the Lorenz equations with standard parameters. Use the known behavior near the cusp, and also use the known behaviour near the fixed point at $z = 27$. (Talk to Dr. Jim about this.) Explore to what extent the function f is symmetric about the cusp. Is this a coincidence, or is there an explanation for the approximate symmetry?

B. Do something like project 12 listed below. Here is a new idea I have for doing this: Compute a numerical sequence of 0s and 1s, where 0 means the loop is on the same side, and 1 means the loop is on the other side. For example RRRLLRRRL-RLRRRLLLR gets mapped to 00101001111001001. Find the fractions of 0s and the fractions of 1s observed in a large numerical sequence. See if you get the same fractions for different runs with different initial conditions, of with different windows of (say) 10,000 symbols in a run or 1,000,000. Check the hypothesis that this map has no "memory." In other words, are these independent observations? Is the observed probability of 0 independent of whether it was preceded by a 0 or by a 1?

A big component of this project is figuring out how to phrase these questions correctly using ideas from statistics. Talk to your colleagues about this!

C. Investigate model 1D maps which are similar to the Lorenz map. Move the cusp to 0, and assume the map is even about 0, and replace the cusp of the form with the $|x|^{.23}$ behaviour to something less severe. For example, investigate maps like $f(x) = a - b\sqrt{|x|}$ or more generally $f(x) = a - b|x|^c$. Find parameters where $f^2(0) = f(a)$ is the left fixed point. For the Lorenz equations with standard parameters, $f^2(0)$ is just to the right of the fixed point.

D. Give a report about the sequence of bifurcations in the Lorenz Equations with $\sigma = 10$, $b = 8/3$, but with the parameter $1 < r < 28$. This is described in Table 7.3.1 in the Robinson chapter about the Lorenz equations. There are two ways these can be described, and it might be a good topic for a group project. 1. Show the solutions in (x, y, z) space, and 2. relate these behaviors to the 1-D Lorenz map. That Lorenz map can be computed for several values of r. Both of these can be done by modifying Dr. Jim's Mathematica Programs.

E. Write a Python program to reproduce Feigenbaum's computation of the a values of the superstable period 2^n orbits.

F. Write a program on a graphing calculator to reproduce Feigenbaum's computation. Research the story.

G. Give a report on the Chaotic water wheel system that can be described by the Lorenz equations. Search the web, or find section 9.1 of Nonlinear Dynamics and Chaos, by Strogatz.

H. Do a report on Sharkovskii's theorem. (I might not get to this in class.)

I. (See project 4 below.) Write a program like Jim's 1985 BBC Basic program that iterates the logistic map. Our website has Mark Haferkamp's Cobweb diagram generator. But this does not reproduce the program. Use exclusive or printing and keep computing new points, and plotting the two associated lines without saving the point, until stopped by a key or the mouse. It would be nice if this ran in a browser, but maybe that is not possible because of the security built into browsers. Maybe it could be a stand-alone program. There is a BBC Basic emulator available on the web, and maybe you could write it using that!

Here is the pseudocode for the guts of the program. The cobweb diagram is a "bit map" with a square of pixels. Each pixel is black or white, and XOR (exclusive or) printing changes black to white and white to black.

`x`, and `delay` are in memory. The cobweb diagram shows $y = x$ and $y = f(x)$

Repeat until terminated by the user.

```
xn = f(x)
Draw line segments from (x,x) to (x, xn) to (xn, xn) using XOR
Do nothing for time interval delay
x = xn
```

Here are some suggestions from previous semesters:

1. Solve the IVP for the velocity of ball moving vertically, assuming that the frictional force is proportional to the speed *squared*. Solve the ODE for the upward velocity v, with any initial condition $v(0) = v_0$. The ODE is

$$\frac{dv}{dt} = -g - kv^2, \text{ if } v \geq 0, \qquad \frac{dv}{dt} = -g + kv^2, \text{ if } v < 0$$

I suggest you define the terminal speed $v_* > 0$ which corresponds to the (negative) terminal velocity $v = -v_*$, and scale the equations to a dimensionless form. Your solution will have different formulas, depending on the cases $v(0) \geq 0$, $-v_* < v(0) < 0$, $v(0) = -v_0$, and $v(0) < -v_0$.

2. Solve the driven washboard equation with a "flat" washboard, i.e. $H = 0$. The equation becomes linear and solvable in closed form.

3. Investigate models of the 1-dimensional map for the Lorenz equation suggested by Lorenz in his paper. See me for ideas on this.

4. Work on Mark Haferkamp's cobweb diagram generator. Here are some possible changes that I would like to see. I do *not* expect you to do more than a few of these.

- Get rid of the "iterations" box. Do not save the x values! See project I above.
- Draw and iterate the cobweb diagram for the n-th iterate of f.
- Simplify the program by focusing on the logistic map and include buttons with many of the interesting a values. Allow a to change with a mouse click without stopping the program.
- Include a button that changes a to $a_\infty + (a - a_\infty)/\delta$. This illustrates Feigenbaum's constant δ.
- Enable a toggle between a slow iteration mode and a fast iteration mode.
- Make the program work well with both phones and larger screens.
- Make a beep with each iterate at a frequency dependent on x. (Toggle this on/off so it doesn't drive the user crazy.)

5. Reproduce Feigenbaum's experiment of finding the a values of the first several susperstable period 2^n points for the logistic map, using a program that could be

written with a programmable calculator. (In other words, no computer algebra is allowed.)

I'm not sure if he used Newton's method or Secant method, but I think Secant method is a lot easier to program so I suggest you use that.

See if you can find out (on the web) details about what Feigenbaum actually did, and what happened to the programmable calculator he used.

6. Explore the three dimensional parameter space for the driven damped pendulum, or the shaken washboard. Use some numerical integration (for example the Mathematica notebooks on the web site) to find solutions. Eliminate transients and look for stable periodic orbits, chaotic orbits, period doubling bifurcations, and multiple stable solutions at the same parameter value. Fix two parameters and vary the third parameter.

In particular, you might want to investigate the Actual Shaken Washboard, that was brought to class. The shape of the washboard is fixed, and the the only parameters we can vary are the friction constant c and the frequency ω. The machine has $L = 3$ inches, $H = 1/8$ inch and $A = 0.9$ inches. The driving period, $2\pi/\omega$ varies from about 0.6 to 1.6 seconds. See if you can find a value of c that shows a sequence of a period doublings as the driving period is increased.

7. Recall how the green ball came to rest in a finite time (See the video for Homework 2, problem 6). This is not consistent with our model that the frictional force is proportional to the velocity. Investigate various models of friction.

8. Learn about the Lagrangian method in classical mechanics. This uses "generalized coordinates" so you do not have to compute components of forces like in usual Newtonian mechanics. This can be applied to several situations. You can choose one or more of these, or some other application.

- A bead sliding on a wire of the shape $z = f(x)$ without friction. Contrast the dynamics with the approximate $m\ddot{x} = -f'(x)mg$ obtained when the slope $f'(x)$ is small.

- A ball rolling on a track $z = f(x)$ without friction. Contrast the dynamics with the approximate $\frac{7}{5}m\ddot{x} = -f'(x)mg$ obtained when the slope $f'(x)$ is small. And explain that factor of $\frac{7}{5}$.

- Investigate the "inverse problem" from the sliding bead. That is, what shape of surface $z = f(x)$ is needed to get a given force function $F(x)$ with dynamics $\ddot{x} = F(x)$. If $F(x) = -kx$, so the period of oscillations is independent of amplitude, this is the famous tautochrone problem, and the shape $y = f(x)$ is a catenoid.

- Investigate the equations of motion of a charged particle in an electric and Magnetic field.

- Investigate the equations of motion in a rotating coordinate system

9. Do a literature search, and/or a numerical investigation of the restricted three-body problem of classical mechanics. Two bodies of Mass M and mass m rotate in a circular orbit about their center of mass. A third body of negligible mass moves in their plane of motion. This can be reduced to a two-degree of freedom Hamiltonian system.

10. The Alligood textbook available on the Cline library website has many "Challenges" that would be good pencil-and-paper projects.

11. Write a program that will plot a histogram of density for the logistic map as a function of the parameter a. Compute the Lyapunov exponent to get a heuristic for how transient iterates to do before collecting data for the histogram. Then, make a movie showing how this histogram changes as a function of the parameter a.

This can be done in Mathematica or MATLAB, but a compiled language would be faster, especially when making the movie.

12. Write a program that computes the Poincaré map for the Lorenz equations with the standard parameters. Use this to get a very long sequence of -1s and 1s, representing L's and R's for an orbit in the attractor starting from some random initial condition. Test the null hypothesis that this is a 2 state Markov process with probability p that an L is followed by an L or an R is followed by an R, and probability $1 - p$ that an L is followed by an R or that an R is followed by an L.

Then process the sequence to get a new sequence of the number of copies of the same letter in a row. For example RRRLLRRRLRLRRRLLLR gets mapped to 2311133. (Do not use the initial RRR and the final R. We do not know how many R's precede the initial R.) Test the hypothesis that this new sequence is iid.

You can get many "realizations" of the sequences by starting with different random initial conditions, or by chopping up a very long sequence into sub-sequences.

You may do any type of statistical analysis on these sequences that makes sense. Don't be constrained by what I suggested here. I have audited STA 570, but taken no other statistics classes. So a big part of this project is to understand what I'm trying to say here and restate it with proper statistical jargon.

13. Do a report on the work of Warwick Tucker, entitled "The Lorenz attractor exists." In the late 1990's he did a rigorous proof Lorenz's 1963 conjecture. You can search for "Lorenz attractor exists" to find his papers on the web. A good place to start on this is with the review article "Mathematics: The Lorenz attractor exists", Nature 406, 948-949 (31 August 2000), by Ian Stewart.